

Analysis III - 203(d)

Winter Semester 2024

Session 11: November 28, 2024

Exercise 1 Compute the Fourier coefficients of the following functions, which have period $T = 2\pi$ and have the given values over the interval $[0, 2\pi)$:

•

$$f(x) = e^{x-\pi}$$

•

$$g(x) = (x - \pi)^3$$

•

$$h(x) = \begin{cases} \sin(x) & 0 \leq x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq x < \frac{3\pi}{2} \\ \sin(x) & \frac{3\pi}{2} \leq x < 2\pi \end{cases}$$

Hint: Recall that $\cos(\frac{\pi}{2} - \theta) = \sin \theta$ and $\cos(\frac{3\pi}{2} - \theta) = -\sin \theta$.

Solution 1 • To compute the Fourier coefficients of $f(x)$ with a period of 2π , use the formulas from the lecture:

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} e^{x-\pi} dx = \frac{1}{2\pi} [e^{x-\pi}]_0^{2\pi} = \frac{1}{2\pi} (e^\pi - e^{-\pi})$$

For other coefficients, we apply integration by parts twice, and see that we obtain the same integral again:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} e^{x-\pi} \cos(nx) dx \\ &= \frac{1}{\pi} \left[e^{x-\pi} \frac{\sin(nx)}{n} \right]_{x=0}^{x=2\pi} - \frac{1}{\pi} \int_0^{2\pi} e^{x-\pi} \frac{\sin(nx)}{n} dx \\ &= \frac{1}{\pi} \left[e^{x-\pi} \frac{\sin(nx)}{n} \right]_{x=0}^{x=2\pi} - \frac{1}{\pi} \left[e^{x-\pi} \frac{-\cos(nx)}{n^2} \right]_{x=0}^{x=2\pi} - \frac{1}{\pi} \int_0^{2\pi} e^{x-\pi} \frac{-\cos(nx)}{n^2} dx \\ &= \frac{1}{\pi} \left(e^\pi \frac{\sin(2\pi)}{n} - e^{0-\pi} \frac{\sin(n0)}{n} \right) - \frac{1}{\pi} \left(e^\pi \frac{-\cos(n2\pi)}{n^2} - e^{-\pi} \frac{-\cos(n0)}{n^2} \right) \\ &\quad + \frac{1}{\pi} \int_0^{2\pi} e^{x-\pi} \frac{-\cos(nx)}{n^2} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi} \frac{e^\pi - e^{-\pi}}{n^2} + \frac{1}{\pi} \int_0^{2\pi} e^{x-\pi} \frac{-\cos(nx)}{n^2} dx \\
&= \frac{1}{\pi} \frac{e^\pi - e^{-\pi}}{n^2} - \frac{1}{\pi} \int_0^{2\pi} e^{x-\pi} \frac{\cos(nx)}{n^2} dx
\end{aligned}$$

We rearrange that

$$\begin{aligned}
&\int_0^{2\pi} e^{x-\pi} \cos(nx) dx = \frac{e^\pi - e^{-\pi}}{n^2} - \frac{1}{n^2} \int_0^{2\pi} e^{x-\pi} \cos(nx) dx \\
\Rightarrow \quad \left(1 + \frac{1}{n^2}\right) \int_0^{2\pi} e^{x-\pi} \cos(nx) dx &= \frac{e^\pi - e^{-\pi}}{n^2} \\
\Rightarrow \quad \int_0^{2\pi} e^{x-\pi} \cos(nx) dx &= \left(1 + \frac{1}{n^2}\right)^{-1} \frac{e^\pi - e^{-\pi}}{n^2}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
a_n &:= \frac{1}{\pi} \int_0^{2\pi} e^{x-\pi} \cos(nx) dx = \left(1 + \frac{1}{n^2}\right)^{-1} \frac{e^\pi - e^{-\pi}}{\pi n^2} \\
&= \left(\frac{n^2 + 1}{n^2}\right)^{-1} \frac{e^\pi - e^{-\pi}}{\pi n^2} \\
&= \frac{n^2}{1 + n^2} \frac{e^\pi - e^{-\pi}}{\pi n^2} \\
&= \frac{e^\pi - e^{-\pi}}{(1 + n^2)\pi}
\end{aligned}$$

The coefficients of the sine modes are:

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_0^{2\pi} e^{x-\pi} \sin(nx) dx \\
&= \frac{1}{\pi} \left[-e^{x-\pi} \frac{\cos(nx)}{n} \right]_{x=0}^{x=2\pi} + \frac{1}{\pi} \int_0^{2\pi} \frac{e^{x-\pi} \cos(nx)}{n} dx \\
&= \frac{1}{\pi} \left[-e^{x-\pi} \frac{\cos(nx)}{n} \right]_{x=0}^{x=2\pi} + \frac{1}{\pi} \left[e^{x-\pi} \frac{\sin(nx)}{n^2} \right]_{x=0}^{x=2\pi} - \frac{1}{\pi} \int_0^{2\pi} e^{x-\pi} \frac{\sin(nx)}{n^2} dx \\
&= \frac{1}{\pi} \frac{e^{-\pi} - e^\pi}{n} - \frac{1}{\pi} \int_0^{2\pi} e^{x-\pi} \frac{\sin(nx)}{n^2} dx
\end{aligned}$$

Similar as above, it follows that

$$b_n := \frac{1}{\pi} \int_0^{2\pi} e^{x-\pi} \sin(nx) dx = \frac{1}{1 + \frac{1}{n^2}} \frac{1}{\pi} \frac{e^{-\pi} - e^\pi}{n} = \frac{1}{\pi} \frac{e^{-\pi} - e^\pi}{n + \frac{1}{n}}$$

- We compute the average

$$\frac{a_0}{2} := \frac{1}{2\pi} \int_0^{2\pi} (x - \pi)^3 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} z^3 dz = 0.$$

We compute the coefficients of the cosine modes: for $n \geq 1$, we find

$$\begin{aligned} a_n &:= \frac{2}{2\pi} \int_0^{2\pi} (x - \pi)^3 \cos(nx) dx \\ &= \frac{1}{\pi} \int_0^{2\pi} (x - \pi)^3 \cos(nx) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} z^3 \cos(n(z + \pi)) dz \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} z^3 \cos(nz + n\pi) dz \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} z^3 \cos(nz) dz \cdot \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}. \end{aligned}$$

However, $z^3 \cos(nz)$ is the product of an odd and an even function, and so the integral vanishes. Explicitly:

$$\begin{aligned} \int_{-\pi}^{\pi} z^3 \cos(nz) dz &= \int_{-\pi}^0 z^3 \cos(nz) dz + \int_0^{\pi} z^3 \cos(nz) dz \\ &= - \int_0^{\pi} w^3 \cos(nw) dw + \int_0^{\pi} z^3 \cos(nz) dz = 0. \end{aligned}$$

Finally, we compute the coefficients of the sine modes:

$$\begin{aligned} b_n &:= \frac{2}{2\pi} \int_0^{2\pi} (x - \pi)^3 \sin(nx) dx \\ &= \frac{1}{\pi} \int_0^{2\pi} (x - \pi)^3 \sin(nx) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} z^3 \sin(n(z + \pi)) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} z^3 \sin(nz) dz \cdot \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}. \end{aligned}$$

In either case, since z^3 and $\sin(nz)$ are odd functions,

$$\int_{-\pi}^{\pi} z^3 \sin(nz) dz = 2 \int_0^{\pi} z^3 \sin(nz) dz.$$

We perform integration by parts several times:

$$\begin{aligned}
& \int_0^\pi z^3 \sin(nz) dz \\
&= -\frac{1}{n} [z^3 \cos(nz)]_{z=0}^{z=\pi} + \frac{3}{n} \int_0^\pi z^2 \cos(nz) dz \\
&= -\frac{1}{n} [z^3 \cos(nz)]_{z=0}^{z=\pi} + \frac{3}{n^2} [z^2 \sin(nz)]_{z=0}^{z=\pi} - \frac{6}{n^2} \int_0^\pi z \sin(nz) dz \\
&= -\frac{1}{n} [z^3 \cos(nz)]_{z=0}^{z=\pi} + \frac{3}{n^2} [z^2 \sin(nz)]_{z=0}^{z=\pi} + \frac{6}{n^3} [z \cos(nz)]_{z=0}^{z=\pi} - \frac{6}{n^3} \int_0^\pi \cos(nz) dz \\
&= -\frac{1}{n} [z^3 \cos(nz)]_{z=0}^{z=\pi} + \frac{3}{n^2} [z^2 \sin(nz)]_{z=0}^{z=\pi} + \frac{6}{n^3} [z \cos(nz)]_{z=0}^{z=\pi} + \frac{6}{n^4} [\sin(nz)]_{z=0}^{z=\pi} \\
&= -\frac{\pi^3(-1)^n}{n} + \frac{6\pi(-1)^n}{n^3} \\
&\implies b_n = \left(\begin{cases} \frac{2\pi^2}{n} - \frac{12}{n^3}, & \text{if } n \text{ is odd,} \\ -\frac{2\pi^2}{n} + \frac{12}{n^3}, & \text{if } n \text{ is even,} \end{cases} \right) \cdot \left(\begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases} \right) = -\frac{2\pi^2}{n} + \frac{12}{n^3}.
\end{aligned}$$

- The average of $h(x)$ over 1 period T is 0 therefore:

$$\frac{a_0}{2} = 0$$

Moreover $h(x)$ is odd therefore $a_n = 0$. Explicitly, we compute

$$a_n = \frac{1}{\pi} \int_0^{2\pi} h(x) \cos(nx) dx = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \sin(x) \cos(nx) dx + \frac{1}{\pi} \int_{\frac{3\pi}{2}}^{2\pi} \sin(x) \cos(nx) dx.$$

To compute the integrals, we apply the product-to-sum identity

$$\sin(x) \cos(nx) = \frac{1}{2} (\sin((1+n)x) + \sin((1-n)x)).$$

We obtain for the first integral

$$\begin{aligned}
& \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \sin(x) \cos(nx) dx = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{2} (\sin((1+n)x) + \sin((1-n)x)) dx \\
&= \frac{1}{\pi} \left(\frac{1}{2} \left[-\frac{1}{1+n} \cos((1+n)x) - \frac{1}{1-n} \cos((1-n)x) \right]_0^{\frac{\pi}{2}} \right) \\
&= -\frac{1}{n+1} \cos\left(\frac{(n+1)\pi}{2}\right) - \frac{1}{1-n} \cos\left(\frac{(1-n)\pi}{2}\right) + \frac{1}{n+1} + \frac{1}{1-n} \\
&= \frac{1}{2\pi(n^2-1)} \left[(1-n) \cos\left(\frac{\pi}{2} + \frac{n\pi}{2}\right) + (n+1) \cos\left(\frac{\pi}{2} - \frac{n\pi}{2}\right) \right] - \frac{1}{\pi(n^2-1)}
\end{aligned}$$

$$= \frac{n \sin\left(\frac{n\pi}{2}\right) - 1}{\pi(n^2 - 1)},$$

where we have applied the hint in the last equation. For the second integral, we obtain

$$\begin{aligned} \frac{1}{\pi} \int_{\frac{3\pi}{2}}^{2\pi} \sin(x) \cos(nx) dx &= \frac{1}{\pi} \int_{\frac{3\pi}{2}}^{2\pi} \frac{1}{2} (\sin((1+n)x) + \sin((1-n)x)) dx \\ &= \frac{1}{\pi} \left(\frac{1}{2} \left[-\frac{1}{1+n} \cos((1+n)x) - \frac{1}{1-n} \cos((1-n)x) \right]_{\frac{3\pi}{2}}^{2\pi} \right) \\ &= \frac{1}{\pi(n^2 - 1)} + \frac{1}{2\pi(1 - n^2)} \left[(1-n) \cos\left((1+n)\frac{3\pi}{2}\right) + (1+n) \cos\left((1-n)\frac{3\pi}{2}\right) \right] \\ &= \frac{1}{\pi(n^2 - 1)} + \frac{1}{2\pi(1 - n^2)} \left[(1-n) \sin\left(\frac{3\pi n}{2}\right) - (1+n) \sin\left(\frac{3\pi n}{2}\right) \right] \\ &= \frac{1 + n \sin\left(\frac{3\pi n}{2}\right)}{\pi(n^2 - 1)}, \end{aligned}$$

again by using the hint. Therefore, we have

$$a_n = \frac{n}{\pi(n^2 - 1)} \left[\sin\left(\frac{\pi n}{2}\right) + \sin\left(\frac{3\pi n}{2}\right) \right] = 0,$$

since the sum-to-product formula for the sine yields

$$\sin\left(\frac{\pi n}{2}\right) + \sin\left(\frac{3\pi n}{2}\right) = 2 \sin(2\pi n) \cos(\pi n) = 0.$$

For b_n we have:

$$\begin{aligned} b_n &= \frac{2}{2\pi} \int_0^{\frac{\pi}{2}} \sin(x) \sin(nx) dx + \frac{2}{2\pi} \int_{\frac{3\pi}{2}}^{2\pi} \sin(x) \sin(nx) dx \\ &= \frac{2}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x) \sin(nx) dx \\ &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x) \sin(nx) dx \\ &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin(x) \sin(nx) dx \\ &= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \cos((n-1)x) - \cos((n+1)x) dx \\ &= \frac{1}{\pi} \left[\frac{\sin((n-1)x)}{n-1} - \frac{\sin((n+1)x)}{n+1} \right]_{x=0}^{x=\frac{\pi}{2}} \end{aligned}$$

$$\begin{aligned}
&= \frac{n \left(\cos\left(\frac{n\pi}{2}\right) + \cos\left(\frac{3n\pi}{2}\right) \right)}{\pi(1-n^2)} \\
&= \frac{2n \cos(n\pi) \cos\left(-\frac{n\pi}{2}\right)}{\pi(1-n^2)} = \frac{2n(-1)^n \cos\left(\frac{n\pi}{2}\right)}{\pi(1-n^2)},
\end{aligned}$$

where we have again used a reflection formula and a product-to-sum formula to simplify the trigonometric expressions. Now we see immediately, that if n is odd, then $b_n = 0$ since $\cos\left(\frac{n\pi}{2}\right) = 0$. Whenever n is even, i.e. $n = 2k$ for some $k \in \mathbb{N}$, we have

$$b_n = b_{2k} = \frac{4k(-1)^k}{\pi(1-4k^2)}$$

Exercise 2 Compute the Fourier coefficients of the function f with period $T = 2\pi$ and which satisfies

$$f(x) = \sin(x) \text{ if } 0 \leq x < \pi,$$

and

$$f(x) = f(-x) \text{ for all } x \in \mathbb{R}.$$

Solution 2 Note that $f(x)$ is an even function. Using a change of variables, we compute

$$\begin{aligned}
b_n &= \frac{2}{T} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \\
&= \frac{2}{T} \int_{-\pi}^0 f(x) \sin(nx) dx + \frac{2}{T} \int_0^{\pi} f(x) \sin(nx) dx \\
&= (-1) \frac{2}{T} \int_{\pi}^0 f(-x) \sin(-nx) dx + \frac{2}{T} \int_0^{\pi} f(x) \sin(nx) dx \\
&= \frac{2}{T} \int_0^{\pi} f(x) \sin(-nx) dx + \frac{2}{T} \int_0^{\pi} f(x) \sin(nx) dx \\
&= -\frac{2}{T} \int_0^{\pi} f(x) \sin(nx) dx + \frac{2}{T} \int_0^{\pi} f(x) \sin(nx) dx = 0.
\end{aligned}$$

Therefore $b_n = 0$. The average is

$$\frac{a_0}{2} = \frac{2}{2\pi} \int_0^{\pi} \sin x dx = \frac{1}{\pi} [-\cos x]_0^{\pi} = \frac{2}{\pi}.$$

The cosine mode coefficients are:

$$\begin{aligned}
a_n &= \frac{2}{T} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\
&= \frac{4}{T} \int_0^{\pi} \sin x \cos(nx) dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{4}{T} \int_0^\pi \frac{1}{2} \sin((1-n)x) + \frac{1}{2} \sin((n+1)x) dx \\
&= \frac{2}{T} \int_0^\pi \sin((1-n)x) + \sin((n+1)x) dx \\
&= \frac{2}{T} \int_0^\pi -\sin((n-1)x) + \sin((n+1)x) dx \\
&= \frac{2}{T} \left[\frac{\cos((n-1)x)}{(n-1)} \right]_{x=0}^{x=\pi} + \frac{2}{T} \left[-\frac{\cos((n+1)x)}{(n+1)} \right]_{x=0}^{x=\pi} \\
&= \frac{2}{T} \left(\frac{\cos((n-1)\pi)}{(n-1)} - \frac{1}{(n-1)} - \frac{\cos((n+1)\pi)}{(n+1)} + \frac{1}{(n+1)} \right) \\
&= \frac{2}{T} \left(\frac{\cos(n\pi - \pi)}{(n-1)} - \frac{1}{(n-1)} - \frac{\cos(n\pi + \pi)}{(n+1)} + \frac{1}{(n+1)} \right) \\
&= \frac{2}{T} \left(-\frac{\cos(n\pi)}{(n-1)} - \frac{1}{(n-1)} + \frac{\cos(n\pi)}{(n+1)} + \frac{1}{(n+1)} \right) \\
&= \frac{2}{T} \left(-\frac{(-1)^n}{(n-1)} - \frac{1}{(n-1)} + \frac{(-1)^n}{(n+1)} + \frac{1}{(n+1)} \right) \\
&= \frac{2}{T} \left(\frac{1 + (-1)^n}{(n+1)} - \frac{1 + (-1)^n}{(n-1)} \right) \\
&= \frac{1}{\pi} \left(\frac{1 + (-1)^n}{(n+1)} - \frac{1 + (-1)^n}{(n-1)} \right).
\end{aligned}$$

Here, if n is odd, the coefficient numerators are $1 + (-1) = 0$, and if n is even, they are both equal 2. Thus:

$$a_n = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ -\frac{2}{\pi(n-1)} + \frac{2}{\pi(n+1)}, & \text{if } n \text{ is even,} \end{cases}$$

Exercise 3 Use Dirichlet's theorem to explain whether the Fourier series converges at $x \in [0, T]$ and to which value:

•

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 2 - x & \text{if } 1 \leq x < 2 \end{cases}, \quad T = 2.$$

•

$$f(x) = \begin{cases} \pi & \text{if } 0 \leq x < 1 \\ e^{x-1} & \text{if } 1 \leq x < 2 \\ \sin(x) & \text{if } 2 \leq x < 3 \end{cases}, \quad T = 3.$$

Here, the functions have the given period T .

Solution 3 • f is continuous and finite on the closed interval $[0, 2]$ therefore the Fourier series, $F_N f$, converges to f when we take the limit of N going to ∞ .

- f is continuous and finite on the closed intervals $[0, 1]$, $[1, 2]$ and $[2, 3]$. However, note that f is not continuous in 0, 1 and 2. Therefore the Fourier series, Ff , converges to:

$$g(x) = \begin{cases} \frac{\sin(3)+\pi}{2} & \text{if } x = 0 \\ \pi & \text{if } 0 < x < 1 \\ \frac{\pi+1}{2} & \text{if } x = 1 \\ e^{x-1} & \text{if } 1 < x < 2 \\ \frac{\sin(2)+e^1}{2} & \text{if } x = 2 \\ \sin(x) & \text{if } 2 < x < 3 \\ \frac{\sin(3)+\pi}{2} & \text{if } x = 3 \end{cases}$$

and we extend that function g to a periodic function with period $T = 3$.

Exercise 4 Give the Fourier series in complex notation, when f has period $T = 2$ and

$$f(x) = x \text{ for } 0 \leq x < 2.$$

Solution 4 We can re-use the results about the sawtooth wave.

$$g(x) = x \text{ for } 0 \leq x < 1 \\ a_0^g = 1, \quad a_n^g = 0, \quad b_n^g = \frac{-1}{\pi n}$$

Moreover we can write f in terms of g by introducing $x = \frac{y}{2}$

$$f(y) = 2g(y/2)$$

which gives us the Fourier coefficients of f

$$a_0^f = 2, \quad a_n^f = 0, \quad b_n^f = \frac{-2}{\pi n}$$

Using this we can obtain the Fourier coefficients c_n :

$$c_0 = \frac{a_0}{2} = 2, \quad c_n = \frac{a_n}{2} - \frac{b_n}{2}i = \frac{1}{n\pi}i, \quad c_{-n} = \frac{a_n}{2} + \frac{b_n}{2}i = -\frac{1}{n\pi}i$$

such that

$$Ff(x) = \sum_{n=-\infty}^{\infty} c_n e^{\pi n x i}$$

Exercise 5 Find the Fourier coefficients of $\cos(x)^8$ with period $T = 2\pi$. Use Parseval's identity to compute $\int_0^{2\pi} \cos(x)^{16} dx$.

Solution 5 We will not use the standard approach in integrating the function f . Instead we try to rewrite f as sum of $\cos(nx)$ using trigonometric formulas:

$$\begin{aligned}\cos^2 \alpha &= \frac{1}{2} + \frac{1}{2} \cos 2\alpha \\ \cos \alpha \cos \beta &= \frac{1}{2} \cos(\alpha - \beta) + \frac{1}{2} \cos(\alpha + \beta)\end{aligned}$$

Using these formulas we can rewrite f as follows:

$$\begin{aligned}\cos^8 x &= (\cos^2 x)^4 \\ &= \left(\frac{1}{2} + \frac{1}{2} \cos 2x\right)^4 \\ &= \left(\left(\frac{1}{2} + \frac{1}{2} \cos 2x\right)^2\right)^2 \\ &= \left(\frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x\right)^2 \\ &= \left(\frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x\right)^2 \\ &= \left(\frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x\right)\left(\frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x\right) \\ &= \frac{1}{16} + \frac{1}{4} \cos 2x + \frac{3}{8} \cos^2 2x + \frac{1}{4} \cos^3 2x + \frac{1}{16} \cos^4 2x \\ &= \frac{1}{16} + \frac{1}{4} \cos 2x + \frac{3}{8} \left(\frac{1}{2} + \frac{1}{2} \cos 4x\right) + \frac{1}{4} \cos^3 2x + \frac{1}{16} \cos^4 2x \\ &= \frac{4}{16} + \frac{1}{4} \cos 2x + \frac{3}{16} \cos 4x + \frac{1}{4} \cos^3 2x + \frac{1}{16} \cos^4 2x\end{aligned}$$

Now we consider $\cos^3 2x$ and $\cos^4 2x$ separately:

$$\begin{aligned}\cos^3 2x &= \cos 2x \cos^2 2x \\ &= \left(\frac{1}{2} + \frac{1}{2} \cos 4x\right) \cos 2x \\ &= \frac{1}{2} \cos 2x + \frac{1}{2} \cos 4x \cos 2x \\ &= \frac{1}{2} \cos 2x + \frac{1}{4} \cos 6x + \frac{1}{4} \cos 2x \\ &= \frac{3}{4} \cos 2x + \frac{1}{4} \cos 6x\end{aligned}$$

$$\cos^4 2x = (\cos^2 2x)^2$$

$$\begin{aligned}
&= \left(\frac{1}{2} + \frac{1}{2} \cos 4x\right)^2 \\
&= \frac{1}{4} + \frac{1}{2} \cos 4x + \frac{1}{4} \cos^2 4x \\
&= \frac{1}{4} + \frac{1}{2} \cos 4x + \frac{1}{4} \left(\frac{1}{2} + \frac{1}{2} \cos 8x\right) \\
&= \frac{3}{8} + \frac{1}{2} \cos 4x + \frac{1}{8} \cos 8x
\end{aligned}$$

putting everything together we have:

$$\cos^8 x = \frac{35}{128} + \frac{7}{16} \cos 2x + \frac{7}{32} \cos 4x + \frac{1}{16} \cos 6x + \frac{1}{128} \cos 8x$$

For the second part we use Parseval's identity which is given by:

$$\frac{2}{T} \int_0^T f(x)^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n)^2 + (b_n)^2$$

substituting our Fourier coefficients and f gives:

$$\begin{aligned}
\frac{2}{T} \int_0^T f(x)^2 dx &= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n)^2 + (b_n)^2 \\
\frac{1}{\pi} \int_0^{2\pi} \cos(x)^{16} dx &= 2 \left(\frac{35}{128}\right)^2 + \left(\frac{7}{16}\right)^2 + \left(\frac{7}{32}\right)^2 + \left(\frac{1}{16}\right)^2 + \left(\frac{1}{128}\right)^2 \\
\Rightarrow \int_0^{2\pi} \cos(x)^{16} dx &= \pi \left(2 \left(\frac{35}{128}\right)^2 + \left(\frac{7}{16}\right)^2 + \left(\frac{7}{32}\right)^2 + \left(\frac{1}{16}\right)^2 + \left(\frac{1}{128}\right)^2\right)
\end{aligned}$$

We simplify this value:

$$2 \left(\frac{35}{128}\right)^2 + \left(\frac{7}{16}\right)^2 + \left(\frac{7}{32}\right)^2 + \left(\frac{1}{16}\right)^2 + \left(\frac{1}{128}\right)^2 = 6435$$

Therefore, in total,

$$\int_0^{2\pi} \cos(x)^{16} dx = \pi \frac{6435}{16384} \approx 1.233895796$$

Exercise 6 Give the Fourier series of the function f with period $T = 2$ and

$$f(x) = \cos(x) \text{ for } -1 \leq x < 1.$$

Give the Fourier series in standard form and in complex notation. Compare $F_3 f$ at the points $x = -\pi/4, 0, \pi/4$ with the original function f .

Solution 6 We calculate the complex Fourier coefficients

$$c_0 = \frac{1}{2} \int_{-1}^1 \cos x dx = \frac{1}{2} [\sin x]_{-1}^1 = \sin 1$$

$$\begin{aligned} c_n &= \frac{1}{2} \int_{-1}^1 \cos x e^{-\pi n x i} dx \\ &= \frac{1}{2} \int_{-1}^1 \left(\frac{1}{2} e^{xi} + \frac{1}{2} e^{-xi} \right) e^{-\pi n x i} dx \\ &= \frac{1}{2} \int_{-1}^1 \left(\frac{1}{2} e^{(1-\pi n)x i} + \frac{1}{2} e^{-(1+\pi n)x i} \right) dx \\ &= \frac{1}{2} \left[\frac{1}{(1-\pi n)i} \frac{1}{2} e^{(1-\pi n)x i} - \frac{1}{(1+\pi n)i} \frac{1}{2} e^{-(1+\pi n)x i} \right]_{-1}^1 \\ &= \frac{1}{2(1-\pi n)i} \left(\frac{1}{2} e^{(1-\pi n)i} - \frac{1}{2} e^{-(1-\pi n)i} \right) - \frac{1}{2(1+\pi n)i} \left(\frac{1}{2} e^{(1+\pi n)i} + \frac{1}{2} e^{-(1+\pi n)i} \right) \\ &= \frac{1}{2(1-\pi n)} \sin(1-\pi n) + \frac{1}{2(1+\pi n)} \sin(1+\pi n) \end{aligned}$$

Using c_n we can determine a_n and b_n

$$\frac{a_0}{2} = c_0 = \sin 1, \quad a_n = c_n + c_{-n} = \frac{1}{(1-\pi n)} \sin(1-\pi n) + \frac{1}{(1+\pi n)} \sin(1+\pi n), \quad b_n = i(c_{-n} - c_n) = 0$$

The Fourier series in standard and in complex notation are given by:

$$\begin{aligned} Ff(x) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2(1-\pi n)} \sin(1-\pi n) + \frac{1}{2(1+\pi n)} \sin(1+\pi n) \right) e^{\frac{2\pi n x}{T} i} \\ Ff(x) &= \sin(1) + \sum_{n=1}^{\infty} \left(\frac{1}{(1-\pi n)} \sin(1-\pi n) + \frac{1}{(1+\pi n)} \sin(1+\pi n) \right) \cos(\pi n x) \end{aligned}$$

We compare $F_3 f(x)$ and $f(x)$ at several points

$$\begin{aligned} F_3 f(0) &= 1.00664, & F_3 f\left(-\frac{\pi}{4}\right) &\approx 0.69196, & F_3 f\left(\frac{\pi}{4}\right) &\approx 0.69196, \\ f(0) &= 1, & f\left(-\frac{\pi}{4}\right) &\approx 0.70711, & f\left(\frac{\pi}{4}\right) &\approx 0.70711. \end{aligned}$$