Analysis III - 203(d)

Winter Semester 2024

Session 8: November 7, 2025

Exercise 1 *Consider the surface*

$$S := \left\{\vec{x} \in \mathbb{R}^3 \;\middle|\; \|x\| = 1, x_3 > 0\right\}$$

- *Find a parameterization of S and find the unit normal corresponding to that parameterization.*
- *Find a parameterization of $C = \partial S$.*
- *Verify Stokes theorem with the vector field*

$$\vec{F}(x_1, x_2, x_3) = (-x_2, x_1, x_3).$$

Solution 1 *We conduct this in several steps.*

- *One possible parameterisation of the surface S reads:*

$$\Omega := [0, 2\pi) \times [0, \tfrac{\pi}{2}),$$
$$\Phi : \Omega \to S, \quad (\theta, \phi) \mapsto (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi).$$

We think of this as a parameterization of the upper hemisphere. The first variable θ parameterizes the equator-like lines (latitude) and the second variable paramterizes the longitude (how far up north, starting at the equator). We compute the partial derivatives:

$$\Phi = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi \end{pmatrix}, \quad \partial_\theta\Phi = \begin{pmatrix} -\sin\theta\cos\phi \\ \cos\theta\cos\phi \\ 0 \end{pmatrix}, \quad \partial_\phi\Phi = \begin{pmatrix} -\cos\theta\sin\phi \\ -\sin\theta\sin\phi \\ \cos\phi \end{pmatrix},$$
$$\partial_\theta\Phi \times \partial_\phi\Phi = \begin{pmatrix} \cos\theta\cos^2\phi \\ \sin\theta\cos^2\phi \\ \sin\phi\cos\phi \end{pmatrix}.$$

The norm of this cross product is

$$\begin{aligned} \|\partial_\theta\Phi \times \partial_\phi\Phi\| &= \sqrt{\cos^2\theta\cos^4\phi + \sin^2\theta\cos^4\phi + \sin^2\phi\cos^2\phi} \\ &= \sqrt{\cos^4\phi + \sin^2\phi\cos^2\phi} = \cos\phi\sqrt{\cos^2\phi + \sin^2\phi} = \cos\phi. \end{aligned}$$

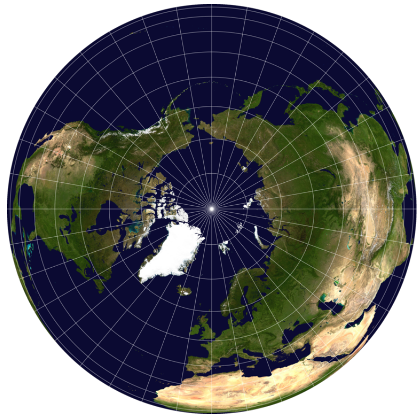


Figure 1: The first variable eastwards. The second variable points up north. The resulting normal is pointing away from the origin. Remark: if we only consider the upper hemisphere surface, then there is no outside or inside pointing normal. Indeed, if you cut open the Earth and only consider the upper hemisphere, then there would be inside left.

Hence

$$\vec{n} = \frac{\partial_\theta \Phi \times \partial_\phi \Phi}{\|\partial_\theta \Phi \times \partial_\phi \Phi\|} = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi \end{pmatrix}$$

According to the right-hand rule, when walking along the boundary with our right hand pointing into the first direction and our left-hand pointing into the interior of the surface, the resulting normal points will be the one pointing away from the origin.

- *We first parameterize the boundary of the parameter space* Ω

$$\gamma : [0, 2\pi) \mapsto \partial\Omega, \quad \theta \mapsto \left(\theta, \frac{\pi}{2}\right),$$

which gives us the parameterization of the boundary as follows:

$$\Phi \circ \gamma : [0, 2\pi) \mapsto \partial C, \quad \theta \mapsto (\cos\theta, \sin\theta, 0).$$

- *We have to show that:*

$$\iint_S \nabla \times \vec{F} d\sigma = \oint_C \vec{F} d\ell$$

On the one hand,

$$\iint_S \nabla \times \vec{F} d\sigma = \iint_S \nabla \times \vec{F} \cdot \vec{n} \|\partial_\theta \Phi \times \partial_\phi \Phi\| \; d\sigma$$

$$
\begin{aligned}
&= \iint_S \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \cos\phi \end{pmatrix} \sin\phi \ d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 2\cos\phi\sin\phi \ d\phi d\theta \\
&= 2\pi \int_0^{\frac{\pi}{2}} 2\cos\phi\sin\phi \ d\phi d\theta = 2\pi \int_0^{\frac{\pi}{2}} \sin(2\phi) \ d\phi d\theta = 2\pi \left[\frac{1}{2}\cos 2\phi\right]_0^{\frac{\pi}{2}} = 2\pi.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\oint_C \vec{F} d\ell &= \int_0^{2\pi} \vec{F}(\Phi\circ\gamma) \cdot \frac{d(\Phi\circ\gamma)}{d\theta} d\theta \\
&= \int_0^{2\pi} \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} d\theta = \int_0^{2\pi} 1 d\theta = 2\pi
\end{aligned}
$$

Exercise 2 *Let S be the surface with parameterization $\Phi : \Omega \to S$ as follows:*

$$
S := \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 = 1, \ -1 < x_3 < 1 \right\},
$$

Suppose we have a vector field

$$
\vec{F}(x_1, x_2, x_3) := (x_2x_3, -x_1x_3, 0).
$$

- *Find a parameterization of the unit normal $\vec{n}$ given associated with that parameterization.*
- *Compute the scalar field $\vec{F} \cdot \vec{n}$ and calculate the integrals of this scalar fields over S.*
- *Compute the integral using the formula*

  $$
  \int_\Omega \vec{F} \cdot (\partial_s\Phi \times \partial_t\Phi) ds dt
  $$

  and compare the result.
- *Find a counterclockwise parameterization of $\partial\Omega$ and use it build a parameterization of the curve $C = \partial S$.*
- *Compute the integrals*

  $$
  \oint_C \vec{F} dl.
  $$

Solution 2

- *In a similar fashion as Exercise 1 Subquestion 2 we obtain the following normal vector:*

$$\vec{n} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

- $$\vec{F} \cdot \vec{n} = \begin{pmatrix} x_2 x_3 \\ -x_1 x_3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = x_1 x_2 x_3 - x_1 x_2 x_3 = 0$$

$$\int_S \vec{F} \cdot \vec{n} d\sigma = \int_S 0 d\sigma = 0$$

- *We define the parameterisation:*

$$\Phi : [0, 2\pi) \times [-1, 1] \mapsto S, (\theta, z) \mapsto (\cos\theta, \sin\theta, z)$$

$$\begin{aligned} \int_\Omega \vec{F}(\Phi) \cdot (\partial_S \Phi \times \partial_t \Phi) dsdt &= \int_0^{2\pi} \int_{-1}^{1} \begin{pmatrix} z\sin\theta \\ -z\cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} dsdt \\ &= \int_0^{2\pi} \int_{-1}^{1} 0 dsdt = 0 \end{aligned}$$

- *A counter clockwise parameterisation of $\partial\Omega$ is given by:*

$$\begin{aligned} \gamma_1 &: [0, 2\pi) \mapsto \partial\Omega_1, (\theta) \mapsto (\theta, -1) \\ \gamma_2 &: [-1, 1) \mapsto \partial\Omega_2, (r) \mapsto (2\pi, r) \\ \gamma_3 &: [0, 2\pi) \mapsto \partial\Omega_3, (\theta) \mapsto (2\pi - \theta, 1) \\ \gamma_4 &: [-1, 1) \mapsto \partial\Omega_4, (r) \mapsto (0, -r) \end{aligned}$$

which we can use to build a parameterization of the curve $C = \partial S$:

$$\begin{aligned} \Phi \circ \gamma_1 &: [0, 2\pi) \mapsto C_1, (\theta) \mapsto (\cos\theta, \sin\theta, -1) \\ \Phi \circ \gamma_2 &: [1, -1) \mapsto C_2, (r) \mapsto (1, 0, r) \\ \Phi \circ \gamma_3 &: [0, 2\pi) \mapsto C_3, (\theta) \mapsto (\cos\theta, -\sin\theta, 1) \\ \Phi \circ \gamma_4 &: [-1, 1) \mapsto C_4, (r) \mapsto (1, 0, -r) \end{aligned}$$

- $$\begin{aligned}
\oint_C \vec{F} dl &= \oint_{C_1 \cup C_2 \cup C_3 \cup C_4} \vec{F} dl \\
&= \sum_{i=1}^{4} \int_{C_i} \vec{F}(\Phi \circ \gamma_i) \cdot (\Phi \circ \gamma_i)' dl \\
&= \int_0^{2\pi} \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} d\theta + \int_{-1}^{1} \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dr + \int_0^{2\pi} \begin{pmatrix} -\sin\theta \\ -\cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ -\cos\theta \\ 0 \end{pmatrix} d\theta + \\
&+ \int_{-1}^{1} \begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dr \\
&= 2 \int_0^{2\pi} \sin^2\theta + \cos^2\theta dr \\
&= 2 \int_0^{2\pi} d\theta = 4\pi
\end{aligned}$$

- *$\vec{F}$ has non-zero curl, so it cannot be a gradient of a scalar field.*

Exercise 3 *Let S be a surface with parameterization $\Phi : \Omega \to S$ as follows:*

$$\begin{aligned}
S &:= \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 > 0 \right\}, \\
\Omega &= \left\{ (s,t) \in \mathbb{R}^2 \;\middle|\; s + t < 1,\ s, t > 0 \right\}, \\
\Phi &: \Omega \to S, \quad (s,t) \mapsto (s, t, 1 - s - t).
\end{aligned}$$

We consider the vector fields

$$\vec{F}(x_1, x_2, x_3) := (x_2 x_3, -x_1 x_3, x_1 x_2), \quad \vec{G}(x_1, x_2, x_3) := (-x_2, x_1, x_3)$$

- *Find the unit normal $\vec{n}$ given associated with that parameterization.*
- *Compute the scalar fields $\vec{F} \cdot \vec{n}$ and $\vec{G} \cdot \vec{n}$ and calculate the integrals of these scalar fields over S.*
- *Compute the integrals using the formula*

  $$\int_\Omega \vec{F} \cdot (\partial_S \Phi \times \partial_t \Phi) ds dt$$

  and compare the result.
- *Find a counterclockwise parameterization of $\partial\Omega$ and use it build a parameterization of the curve $C = \partial S$.*

- *Compute the integrals*

$$\oint_C \vec{F} dl, \quad \oint_C \vec{G} dl.$$

- *Can you exclude that $\vec{F}$ or $\vec{G}$ are gradients of a scalar field?*

Solution 3

- *In a similar fashion as exercise 1 subquestion 2 we obtain the following normal vector:*

$$\vec{n} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- $$\begin{aligned} \int_\Omega \vec{F}(\Phi) \cdot (\partial_S \Phi \times \partial_t \Phi) dsdt &= \int_\Omega \vec{F}(\Phi) \cdot (\partial_S \Phi \times \partial_t \Phi) \frac{\|\partial_S \Phi \times \partial_t \Phi\|}{\|\partial_S \Phi \times \partial_t \Phi\|} dsdt \\ &= \int_\Omega \vec{F}(\Phi) \cdot \vec{n} \|\partial_S \Phi \times \partial_t \Phi\| dsdt \\ &= \int_S \vec{F} \cdot \vec{n} d\sigma \end{aligned}$$

- $$\partial_s \Phi \times \partial_t \Phi = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \int_S \vec{F}(\Phi) \cdot (\partial_S \Phi \times \partial_t \Phi) dsdt &= \int_0^1 \int_0^{1-t} \begin{pmatrix} t(1-s-t) \\ -s(1-s-t) \\ st \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dsdt \\ &= \int_0^1 \int_0^{1-t} t(1-s-t) - s(1-s-t) + stdsdt \end{aligned}$$

$$\int_0^1 \int_0^{1-t} t(1-s-t) dsdt = \int_0^1 \frac{1}{2}t - t^2 + \frac{1}{2}t^3 dt = \frac{1}{24}$$
$$\int_0^1 \int_0^{1-t} s(1-s-t) dsdt = \int_0^1 \int_0^{1-t} \frac{1}{2}(1-s-t)^2 dsdt = \int_0^1 \frac{1}{6}(1-t)^3 dt = \frac{1}{24}$$

$$\int_0^1 \int_0^{1-t} stdsdt = \int_0^1 \frac{1}{2}(1-t)^2 tdt = \int_0^1 \frac{1}{6}(1-t)^3 dt = \frac{1}{24}$$

$$\int_0^1 \int_0^{1-t} t(1-s-t) - s(1-s-t) + stdsdt = \frac{1}{24} - \frac{1}{24} + \frac{1}{24} = \frac{1}{24}$$

$$\begin{aligned}
\int_S \vec{G}(\Phi) \cdot (\partial_S \Phi \times \partial_t \Phi) dsdt &= \int_0^1 \int_0^{1-t} \begin{pmatrix} -t \\ s \\ 1-s-t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dsdt \\
&= \int_0^1 \int_0^{1-t} 1 - 2tdsdt \\
&= \int_0^1 (1-2t)(1-t)dt \\
&= \left[-\frac{1}{2}(1-2t)(1-t) \right]_0^1 - \int_0^1 (1-t)^2 dt \\
&= \frac{1}{2} - \left[-\frac{1}{3}(1-t)^3 \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}
\end{aligned}$$

- *The boundary of Ω consists of three lines which can be parameterized in a counter clockwise way as follows:*

$$\begin{aligned}
\gamma_1 &: [0,1] \mapsto \partial\Omega_1, (r) \mapsto (r, 0), \\
\gamma_2 &: [0,1] \mapsto \partial\Omega_2, (r) \mapsto (1-r, r), \\
\gamma_3 &: [0,1] \mapsto \partial\Omega_3, (r) \mapsto (0, 1-r),
\end{aligned}$$

which we can use to parameterize the curve C which consist of three lines:

$$\begin{aligned}
\Phi \circ \gamma_1 &: [0,1] \mapsto C_1, (r) \mapsto (r, 0, 1-r), \\
\Phi \circ \gamma_2 &: [0,1] \mapsto C_2, (r) \mapsto (1-r, r, 0), \\
\Phi \circ \gamma_3 &: [0,1] \mapsto C_3, (r) \mapsto (0, 1-r, r),
\end{aligned}$$

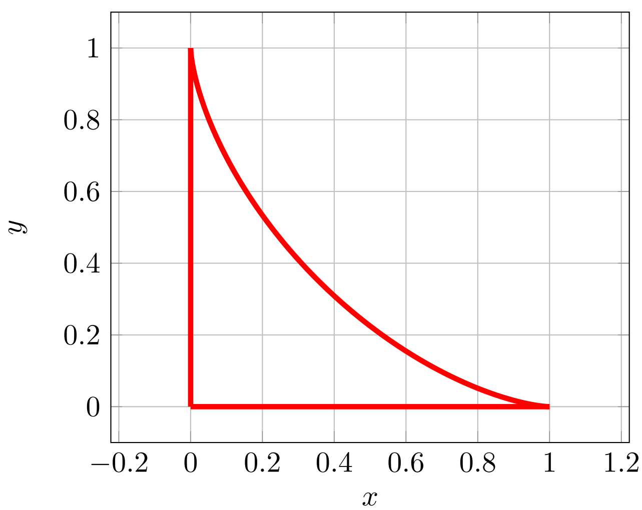
- $$\oint_C \vec{F} dl = \oint_{C_1 \cup C_2 \cup C_3} \vec{F} dl$$

$$
\begin{aligned}
&= \int_{C_1} \vec{F}(\Phi\circ\gamma_1)\cdot\frac{d(\Phi\circ\gamma_1)}{dr}dl + \int_{C_2} \vec{F}(\Phi\circ\gamma_2)\cdot\frac{d(\Phi\circ\gamma_2)}{dr}dl + \int_{C_3} \vec{F}(\Phi\circ\gamma_3)\cdot\frac{d(\Phi\circ\gamma_3)}{dr}dl \\
&= \int_0^1 \begin{pmatrix}0\\-r+r^2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}dr + \int_0^1 \begin{pmatrix}0\\0\\r-r^2\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix}dr + \int_0^1 \begin{pmatrix}r-r^2\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\1\end{pmatrix}dr = \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\oint_C \vec{G}dl &= \oint_{C_1\cup C_2\cup C_3} \vec{G}dl \\
&= \int_{C_1} \vec{G}(\Phi\circ\gamma_1)\cdot\frac{d(\Phi\circ\gamma_1)}{dr}dl + \int_{C_2} \vec{G}\Phi\circ\gamma_2)\cdot\frac{d(\Phi\circ\gamma_2)}{dr}dl + \int_{C_3} \vec{G}(\Phi\circ\gamma_3)\cdot\frac{d(\Phi\circ\gamma_3)}{dr}dl \\
&= \int_0^1 \begin{pmatrix}0\\r\\1-r\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}dr + \int_0^1 \begin{pmatrix}-r\\1-r\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix}dr + \int_0^1 \begin{pmatrix}r-1\\0\\r\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\1\end{pmatrix}dr = \\
&= \int_0^1 r-1dr + \int_0^1 r+1-rdr + \int_0^1 rdr \\
&= \int_0^1 2rdr = 1
\end{aligned}
$$

- *Both $\vec{F}$ and $\vec{G}$ have non-zero curl, so they cannot be gradients of a scalar field.*

Exercise 4 (Green's theorem) *Consider the following closed curve C:*



The curved part of the curve can be parameterized by

$$
\gamma : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}^2, \quad t \mapsto \left(\cos^3(t), \sin^3(t)\right).
$$

- *Find parameterizations of the other two curves.*
- *With the* $\vec{F}(x_1, x_2) = (x_2, -x_1)$ *compute the curve integral*

$$\oint_C \vec{F} dl$$

- *Use Green's theorem to explain how to compute the size of the area enclosed by the curve* C.

Solution 4

-

$$\gamma_1 : [0,1] \to \mathbb{R}^2, \quad t \mapsto (0, 1-t)\,.$$

$$\gamma_2 : [0,1] \to \mathbb{R}^2, \quad t \mapsto (t, 0)\,.$$

-

$$\begin{aligned}\oint_C \vec{F} dl &= \int_0^1 \begin{pmatrix} 1-t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt + \int_0^1 \begin{pmatrix} 0 \\ -t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \int_0^{\frac{\pi}{2}} \begin{pmatrix} \sin^3 t \\ -\cos^3 t \end{pmatrix} \cdot \begin{pmatrix} -3\cos^2 t \sin t \\ 3 \sin^2 t \cos t \end{pmatrix} dt \\ &= \int_0^{\frac{\pi}{2}} -3\cos^2 t \sin^2 t dt \\ &= \int_0^{\frac{\pi}{2}} -\frac{3}{4} \sin^2 2t dt \\ &= \int_0^{\frac{\pi}{2}} \frac{3}{4} \left(\frac{1}{2} \cos 4t - \frac{1}{2} \right) dt \\ &= \int_0^{\frac{\pi}{2}} -\frac{3}{8} dt = -\frac{3\pi}{16}\end{aligned}$$

- *We want to evaluate:*

$$\iint_\Omega 1 dx dy$$

and Green's theorem tells us that:

$$\oint_C \vec{F} dl = \iint_\Omega \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} dx_1 dx_2$$

In the previous exercise we solved the line integral with the vector field $\vec{F}(x_1, x_2) = (x_2, -x_1)$. *If we apply Greens theorem for this vector field then*

$$-\frac{3\pi}{16} = \oint_C \vec{F} dl = \iint_\Omega \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} dx_1 dx_2 = \iint_\Omega -2 dx_1 dx_2$$

Therefore we have that

$$-\frac{3\pi}{16} = \iint_\Omega -2 dx_1 dx_2$$

and consequently

$$\frac{3\pi}{32} = \iint_\Omega dx_1 dx_2$$

Exercise 5 (Stokes's theorem) *We have a vector field*

$$\vec{F}(x_1, x_2, x_3) = (x_1 x_2, x_2 x_3, x_1 x_3)$$

and a surface

$$S := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 = 1,\ x_2 \geq 0, x_3 \geq 0 \right\}$$

- *Find parameterizations of the surface S and a parameterization of its boundary curve C.*
- *Use Stokes' theorem to compute the integral*

$$\iint_S \operatorname{curl} \vec{F}\, d\sigma.$$

Solution 5 *This is a quarter of a globe. We parameterize this surface by The surface is parameterized by*

$$\begin{gathered} \Omega := [0, \tfrac{\pi}{2}] \times [0, \pi], \\ \Phi : \Omega \mapsto S, \quad (\theta, \phi) \mapsto (\cos\phi, \cos\theta \sin\phi, \sin\theta \sin\phi) \end{gathered}$$

For the boundary parameterisation, we walk from the North pole to South pole along one side and then back from the South pole to the North pole along the other side. The parameterization is given by the two curves

$$\begin{aligned} \gamma_1 &: [0, \pi] \to \mathbb{R}^3, \quad \phi \mapsto (\cos\phi, \sin\phi, 0), \\ \gamma_2 &: [0, \pi] \to \mathbb{R}^3, \quad \phi \mapsto (\cos(\pi - \phi), 0, \sin(\pi - \phi)). \end{aligned}$$

The derivatives of those two curves are

$$\begin{aligned}
\dot\gamma_1 &: [0,\pi] \to \mathbb{R}^3, \quad \phi \mapsto (-\sin\phi, \cos\phi, 0),\\
\dot\gamma_2 &: [0,\pi] \to \mathbb{R}^3, \quad \phi \mapsto (\sin(\pi-\phi), 0, -\cos(\pi-\phi)).
\end{aligned}$$

Equipped with that, we use the vector field as in the problem statement,

$$\vec{F}(x_1,x_2,x_3) = (x_1x_2, x_2x_3, x_1x_3)\,,$$

and, after renaming the paramter variable from ϕ to t, compute

$$\begin{aligned}
\iint_S \operatorname{curl}\vec{F}\; d\sigma &= \iint_{\gamma_1} \left(\vec{F}\circ\gamma_1\right)\cdot\dot\gamma_1\; d\sigma + \iint_{\gamma_2} \left(\vec{F}\circ\gamma_2\right)\cdot\dot\gamma_2\; d\sigma\\
&= \int_0^\pi \begin{pmatrix}\cos t\sin t\\ 0\\ 0\end{pmatrix}\cdot\begin{pmatrix}-\sin t\\ \cos t\\ 0\end{pmatrix} dt + \int_0^\pi \begin{pmatrix}0\\ 0\\ \sin(\pi-t)\cos(\pi-t)\end{pmatrix}\cdot\begin{pmatrix}\sin(\pi-t)\\ 0\\ -\cos(\pi-t)\end{pmatrix} dt\\
&= \int_0^\pi \begin{pmatrix}\cos t\sin t\\ 0\\ 0\end{pmatrix}\cdot\begin{pmatrix}-\sin t\\ \cos t\\ 0\end{pmatrix} dt + \int_0^\pi \begin{pmatrix}0\\ 0\\ \sin(t)\cos(t)\end{pmatrix}\cdot\begin{pmatrix}\sin(t)\\ 0\\ -\cos(t)\end{pmatrix} dt\\
&= \int_0^\pi -\cos t\sin^2 t dt - \int_0^\pi \sin t\cos^2 t dt\\
&= -\left[\tfrac{1}{3}\sin^3(t)\right]_{t=0}^{t=\pi} - \left[\tfrac{1}{3}\cos^3(t)\right]_{t=0}^{t=\pi} = -\left[\tfrac{1}{3}\cos^3(t)\right]_{t=0}^{t=\pi} = -\frac{2}{3}
\end{aligned}$$

This completes the discussion.