Analysis III - 203(d)

Winter Semester 2024

Session 7: October 31, 2024

Exercise 1 *Verify the divergence theorem for the following vector field $\vec{F}$ and volume V:*

$$\vec{F}(x_1, x_2, x_3) := (x_1 x_3, x_2, x_2), \qquad V := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 < 1 \right\}.$$

Note that V is just the three-dimensional unit ball.

Solution 1 *We have to show that the followig identity holds:*

$$\iiint_V \nabla \cdot \vec{F} \, dV = \oint \vec{F} \cdot \vec{n} \, d(\partial V)$$

We start with the volume integral:

$$\begin{aligned}
&\iiint_V \nabla \cdot \vec{F} \, dV \\
&= \iiint_V x_3 + 1 \, dV \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^1 (r\cos\phi + 1) r^2 \sin\phi \, dr \, d\phi \, d\theta \\
&= 2\pi \int_0^1 r^3 \, dr \int_0^{\pi} \cos\phi \sin\phi \, d\phi + 2\pi \int_0^1 r^2 \, dr \int_0^{\pi} \sin\phi \, d\phi \\
&= 2\pi \left[\frac{1}{4} r^4\right]_0^1 \left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi} + 2\pi \left[\frac{1}{3} r^3\right]_0^1 \left[-\cos\phi\right]_0^{\pi} \\
&= 2\pi \left(\frac{1}{4} - 0\right)\left(\frac{1}{4} - -\frac{1}{4}\right) + 2\pi \left(\frac{1}{3} - 0\right)(1 - -1) = \frac{4\pi}{3}.
\end{aligned}$$

Next we consider the surface integral. We need a parameterisation for the surface. One can consider:

$$\Phi : [0, 2\pi) \times (0, \pi) \mapsto S : (\theta, \phi) \mapsto (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

with

$$\partial_\theta \Phi \times \partial_\phi \Phi = \begin{pmatrix} -\sin\theta \sin\phi \\ \cos\theta \sin\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} -\cos\theta\cos\phi \\ \sin\theta \cos\phi \\ -\sin\phi \end{pmatrix} = -\begin{pmatrix} \cos\theta \sin^2\phi \\ \sin\theta \sin^2\phi \\ \sin\phi\cos\phi \end{pmatrix}$$

Note this is inward pointing. Therefore we need an extra minus sign.

$$\oint \vec{F} \cdot \vec{n} \, dS$$

$$= \int_0^{2\pi} \int_0^{\pi} \begin{pmatrix} \cos\theta\sin\phi\cos\phi \\ \sin\theta\sin\phi \\ \sin\theta\sin\phi \end{pmatrix} \cdot \begin{pmatrix} \cos\theta\sin^2\phi \\ \sin\theta\sin^2\phi \\ \sin\phi\cos\phi \end{pmatrix} \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^{\pi} \sin^3\phi\cos\phi \, d\phi + \int_0^{2\pi} \sin^2\theta \, d\theta \int_0^{\pi} \sin^3\phi \, d\phi + \int_0^{2\pi} \sin\theta \, d\theta \int_0^{\pi} \sin^2\phi\cos\phi \, d\phi.$$

The last term is zero due to the fact that the integral of $\sin\theta$ over 1 period is equal to 0. The other integrals are evaluated seperately, where we use the substitution $u = \cos\phi$ several times:

$$\begin{aligned}
&\int_0^{\pi} \sin^3\phi\cos\phi \, d\phi \\
&= \int_0^{\pi} (1-\cos^2\phi)\sin\phi\cos\phi \, d\phi \\
&= \int_0^{\pi} \cos\phi\sin\phi - \cos^3\phi\sin\phi \, d\phi \\
&= \int_0^{\pi} \frac{1}{2}\left[\cos^2(\phi)\right]' + \frac{1}{4}\left[\cos^4(\phi)\right]' \, du \\
&= \frac{1}{2}\left[\cos^2(\phi)\right]_{\phi=0}^{\phi=\pi} + \frac{1}{4}\left[\cos^4(\phi)\right]_{\phi=0}^{\phi=\pi} = 0
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_0^{2\pi} \sin^2\theta \, d\theta \\
&= \int_0^{2\pi} \frac{1}{2}(1-\cos 2\theta) \, d\theta \\
&= \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \pi
\end{aligned}$$

Next,

$$\begin{aligned}
&\int_0^{\pi} \sin^3\phi \, d\phi \\
&= \int_0^{\pi} (1-\cos^2\phi)\sin\phi \, d\phi \\
&= \int_{-1}^{1} 1 - u^2 \, du \\
&= \left[u - \frac{1}{3}u^3\right]_{-1}^{1} = \frac{4}{3}
\end{aligned}$$

If we put everything together we get:

$$\oint \vec{F} \cdot \vec{n} \, d(\partial\Omega)$$

$$= \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^{\pi} \sin^3\phi \cos\phi \, d\phi + \int_0^{2\pi} \sin^2\theta \, d\theta \int_0^{\pi} \sin^3\phi \, d\phi$$
$$= \int_0^{2\pi} \cos^2\theta \, d\theta \cdot 0 + \pi\frac{4}{3} = \frac{4\pi}{3}.$$

We conclude that both integrals give the same value thereby verifying the divergence theorem.

Exercise 2 *Consider the volume*

$$V := \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 < 5 \right\}, \tag{1}$$

- *What is the surface S of this volume?*
- *Find the outward pointing unit normal $\vec{n}$ along the surface S of this volume. Write $\vec{n}$ in terms of (x_1, x_2, x_3) at any point on the surface S.*
- *Find a parameterization of the surface S.*
- *Find a vector field $\vec{F}$ such that $\vec{F} \cdot \vec{n} = x_1^2 + x_2 + x_3$ along the surface S.*
- *Use the divergence theorem to compute*

$$\iint_S x_1^2 + x_2 + x_3 dx_1 dx_2 dx_3. \tag{2}$$

Solution 2

- $S := \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 = 5 \right\}$
- *We can write $G(x_1, x_2, x_3) = -5 + x_1^2 + x_2^2 + x_3^2$. Then since we know that the gradient is normal to the level curves we have that:*

$$\vec{n} = \frac{\nabla G}{||\nabla G||} = \frac{1}{\sqrt{5}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- *A possible parameterization is:*

$$\Phi : [0, 2\pi) \times [0, \pi) \mapsto S, (\theta, \phi) \mapsto (\sqrt{5}\sin\phi\cos\theta, \sqrt{5}\sin\phi\sin\theta, \sqrt{5}\cos\phi)$$

- $\vec{F} = \sqrt{5} \begin{pmatrix} x_1 \\ 1 \\ 1 \end{pmatrix}$
- *We use the divergence theorem:*

$$\iint_S x_1^2 + x_2 + x_3 dx_1 dx_2 dx_3 = \oint_S \vec{F} \cdot \vec{n} d\sigma$$

$$
\begin{aligned}
&= \iiint_V \nabla \cdot F dV \\
&= \iiint_V \sqrt{5} dV = \sqrt{5} \cdot \iiint_V 1 dV = \frac{4}{3}\pi\sqrt{5}^4 = \frac{4}{3}\pi \cdot 25
\end{aligned}
$$

In the last step, we used the formula for the volume of the ball with radius $r = \sqrt{5}$.

Exercise 3 *Consider the volume*

$$
V := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 < x_3 < 1 \right\}
$$

Find the the boundary S of this volume and compute its surface area.

Solution 3 *The boundary of this domain consist of two surfaces:*

$$
\begin{aligned}
S_a &= \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 < 1, x_3 = 1 \right\}, \\
S_b &= \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 = x_3, 0 < x_3 < 1 \right\}
\end{aligned}
$$

We have that $\partial V = \partial\Omega_a \cup \partial V_b$. To calculate the area we need a parameterisation of both surfaces:

$$
\begin{aligned}
\Phi_a &: [0, 2\pi) \times (0, 1) \mapsto S_a : (\theta, r) \mapsto (r\cos\theta, r\sin\theta, 1), \\
\Phi_b &: [0, 2\pi) \times (0, 1) \mapsto S_b : (\theta, z) \mapsto (\sqrt{z}\cos\theta, \sqrt{z}\sin\theta, z).
\end{aligned}
$$

Moreover we have that

$$
\|\partial_\theta \Phi_a \times \partial_r \Phi_a\| = \left\| \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} \right\| = r
$$

and

$$
\|\partial_\theta \Phi_b \times \partial_z \Phi_b\| = \left\| \begin{pmatrix} -\sqrt{z}\sin\theta \\ \sqrt{z}\cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2\sqrt{z}}\cos\theta \\ \frac{1}{2\sqrt{z}}\sin\theta \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} \sqrt{z}\sin\theta \\ \sqrt{z}\cos\theta \\ -\frac{1}{2} \end{pmatrix} \right\| = \sqrt{z + \frac{1}{4}}
$$

Now the area is given by:

$$
\begin{aligned}
\iint_{\partial\Omega} 1 \, d(\partial\Omega) &= \iint_{\partial\Omega_a} 1 \, dS_a + \iint_{\partial\Omega_b} 1 \, dS_b \\
&= \int_0^{2\pi} \int_0^1 r \, dr \, d\theta + \int_0^{2\pi} \int_0^1 \sqrt{z + \frac{1}{4}} \, dz \, d\theta \\
&= 2\pi \left[\frac{1}{2} r^2 \right]_0^1 + 2\pi \left[\frac{2}{3} \left(z + \frac{1}{4} \right)^{\frac{3}{2}} \right]_0^1 \\
&= \pi + \frac{4\pi}{3} \left(\left(\frac{5}{4} \right)^{\frac{3}{2}} - \left(\frac{1}{4} \right)^{\frac{3}{2}} \right) = \pi + \pi \frac{5\sqrt{5} - 1}{6} = 5\pi \frac{1 + \sqrt{5}}{6}.
\end{aligned}
$$

Exercise 4 *Find a regular parameterization $\Phi(s,t)$ of the surface*

$$S := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; 0 < x_3 < 1,\ x_1^2 + x_2^2 = 1 + x_3^2 \right\}$$

Compute cross product $\partial_s \Phi(s,t) \times \partial_t \Phi(s,t)$ and its norm.

Solution 4 *We define the following parameterisation for the surface S:*

$$\Phi : [0, 2\pi) \times (0,1) \mapsto S, \quad (s,t) \mapsto \left(\sqrt{1+t^2}\cos s, \sqrt{1+t^2}\sin s, t \right)$$

$$\partial_s \Phi(s,t) \times \partial_t \Phi(s,t) = \begin{pmatrix} -\sqrt{1+t^2}\sin s \\ \sqrt{1+t^2}\cos s \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{-t}{\sqrt{1+t^2}}\cos s \\ \frac{-t}{\sqrt{1+t^2}}\sin s \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{1+t^2}\cos s \\ \sqrt{1+t^2}\sin s \\ t \end{pmatrix}$$

$$\|\partial_s \Phi(s,t) \times \partial_t \Phi(s,t)\| = \left\| \begin{pmatrix} \sqrt{1+t^2}\cos s \\ \sqrt{1+t^2}\sin s \\ t \end{pmatrix} \right\| = \sqrt{1+2t^2}$$

Exercise 5 *Consider the volume*

$$V := \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; x_1 + x_2 + x_3 < 1,\ x_1, x_2, x_3 > 0 \right\}, \tag{3}$$

Suppose we have a vector field

$$\vec{F}(x_1, x_2, x_3) = \left(x_1^2 x_2, 3x_2^2 x_3, 9x_3^2 x_1 \right) \tag{4}$$

Use the divergence theorem to compute the surface integral

$$\iint_S \vec{F} d\sigma. \tag{5}$$

Solution 5 *Given any scalar function $g : V \to \mathbb{R}$, we have*

$$\begin{aligned} \oint_S \vec{F} d\sigma &= \iiint_V \nabla \cdot F dV \\ &= \iiint_V 2x_1 x_2 + 6x_2 x_3 + 18x_3 x_1 dV \\ &= \int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} 2x_1 x_2 + 6x_2 x_3 + 18x_3 x_1 dx_1 dx_2 dx_3 \end{aligned}$$

We solve each integral seperately. Note that we apply integration by parts in between steps.

$$\int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} 2x_1 x_2 dx_1 dx_2 dx_3 = \int_0^1 \int_0^{1-x_3} (1 - x_2 - x_3)^2 x_2 dx_2 dx_3$$

$$
\begin{aligned}
&= \int_0^1 \int_0^{1-x_3} \frac{1}{3}(1-x_2-x_3)^3 dx_2 dx_3 \\
&= \int_0^1 \frac{1}{12}(1-x_3)^4 dx_3 \\
&= \left[\frac{-1}{60}(1-x_3)^5\right]_0^1 = \frac{1}{60}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} 6x_2x_3 dx_1 dx_2 dx_3 &= \int_0^1 \int_0^{1-x_3} 6(1-x_2-x_3)x_2x_3 dx_2 dx_3 \\
&= \int_0^1 \int_0^{1-x_3} 3(1-x_2-x_3)^2 x_2x_3 dx_2 dx_3 \\
&= \int_0^1 (1-x_3)^3 x_3 dx_3 \\
&= \int_0^1 \frac{1}{4}(1-x_3)^4 dx_3 \\
&= \left[-\frac{1}{20}(1-x_3)^5\right]_0^1 = \frac{1}{20}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} 18x_3x_1 dx_1 dx_2 dx_3 &= \int_0^1 \int_0^{1-x_3} 9x_3(1-x_2-x_3)^2 dx_2 dx_3 \\
&= \int_0^1 3x_3(1-x_3)^3 dx_3 \\
&= \int_0^1 \frac{3}{4}(1-x_3)^4 dx_3 \\
&= \left[-\frac{3}{20}(1-x_3)^5\right]_0^1 = \frac{3}{20}
\end{aligned}
$$

We conclude that:

$$
\int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} 2x_1x_2 + 6x_2x_3 + 18x_3x_1 dx_1 dx_2 dx_3 = \frac{1}{60} + \frac{1}{20} + \frac{3}{20} = \frac{13}{60}
$$

Exercise 6 *Given the curve*

$$
\gamma : [0, \pi] \to \mathbb{R}^2, \quad t \mapsto (3\cos(2t), \sin(2t))
$$

and a function

$$
f : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto \left(x_1^2 + 81x_2^2\right)^{\frac{3}{2}},
$$

compute the integrals

$$
\int_\Gamma f \, d\ell, \quad \int_\Gamma \nabla f \, d\ell.
$$

Solution 6 *The integral of the gradient along this simple closed regular curve is zero. With regard to the integral of the scalar field along the curve, we have*

$$\dot{\gamma}(t) = (-6\sin(2t), 2\cos(2t)),$$
$$|\dot{\gamma}(t)| = \sqrt{36\sin^2(2t) + 4\cos^2(2t)} = 2\sqrt{9\sin^2(2t) + \cos^2(2t)}.$$

We also see that

$$\begin{aligned} f(\gamma(t)) = \left(9\cos^2(2t) + 81\sin^2(2t)\right)^{\frac{3}{2}} &= 9^{\frac{3}{2}}\left(\cos^2(2t) + 9\sin^2(2t)\right)^{\frac{3}{2}} \\ &= 27\left(\cos^2(2t) + 9\sin^2(2t)\right)^{\frac{3}{2}}, \end{aligned}$$

In combination,

$$\begin{aligned} \int_\Gamma f\,d\ell &= \int_0^\pi f(\gamma(t)) \cdot |\dot{\gamma}(t)|\,dt \\ &= \int_0^\pi 27\left(\cos^2(2t) + 9\sin^2(2t)\right)^{\frac{3}{2}} \cdot 2\sqrt{\cos^2(2t) + 9\sin^2(2t)}\,dt \\ &= 54\int_0^\pi \left(\cos^2(2t) + 9\sin^2(2t)\right)^2\,dt. \end{aligned}$$

Our life will be easier if we substitute $u = 2t$. We also use a trigonometric identity. Then

$$\begin{aligned} &\int_0^\pi \left(\cos^2(2t) + 9\sin^2(2t)\right)^2\,dt \\ &= \frac{1}{2}\int_0^{2\pi} \left(\cos^2(u) + 9\sin^2(u)\right)^2\,du \\ &= \frac{1}{2}\int_0^{2\pi} \left(1 - \sin^2(u) + 9\sin^2(u)\right)^2\,du \\ &= \frac{1}{2}\int_0^{2\pi} \left(1 + 8\sin^2(u)\right)^2\,du \\ &= \frac{1}{2}\int_0^{2\pi} 1 + 16\sin^2(u) + 64\sin^4(u)\,du \\ &= \int_0^{\pi} 1 + 16\sin^2(u) + 64\sin^4(u)\,du \\ &= \pi + 16\int_0^\pi \sin^2(u)\,du + 64\int_0^\pi \sin^4(u)\,du. \end{aligned}$$

We use the half-angle formula:

$$\sin(u)^2 = \frac{1}{2}\left(1 - \cos(2u)\right).$$

Thus

$$\begin{aligned}\int_0^\pi \sin^2(u)\;du &= \frac{1}{2}\int_0^\pi 1-\cos(2u)\;du\\ &= \frac{\pi}{2} - \int_0^\pi \frac{1}{2}\cos(2u)\;du\\ &= \frac{\pi}{2} - \int_0^\pi \frac{1}{4}(\sin(2u))'\;du\\ &= \frac{\pi}{2} - \frac{1}{4}\left[(\sin(2u))'\right]_0^\pi = \frac{\pi}{2}.\end{aligned}$$

Similarly, using the half-angle formula twice:

$$\begin{aligned}\int_0^\pi \sin^4(u)\;du &= \frac{1}{4}\int_0^\pi \left(1-\cos(2u)\right)^2\;du\\ &= \frac{1}{4}\int_0^\pi 1-2\cos(2u)+\cos(2u)^2\;du\\ &= \frac{1}{4}\int_0^\pi 1-2\cos(2u)+\frac{1}{2}\left(1-\cos(4u)\right)\;du\\ &= \frac{1}{4}\int_0^\pi \frac{3}{2}-2\cos(2u)-\frac{1}{2}\cos(4u)\;du\\ &= \frac{1}{4}\int_0^\pi \frac{3}{2}\;du - \frac{1}{2}\int_0^\pi \cos(2u)\;du - \frac{1}{8}\int_0^\pi \cos(4u)\;du\\ &= \frac{3}{8}\pi.\end{aligned}$$

The integrals of $\cos(2u)$ *and* $\cos(4u)$ *from* 0 *to* π *vanish, by the the same argument as above. In total,*

$$\int_\Gamma f d\ell = 54\left(\pi + 16\cdot\frac{\pi}{2} + 64\cdot\frac{3}{8}\pi\right) = 54\left(\pi+8\pi+24\pi\right) = 54\cdot 33\pi = 1782\pi.$$