

Analysis III - 203(d)

Winter Semester 2024

Session 5: October 10, 2025

Exercise 1 Consider a curve u in one-dimensional space with

$$u : [1, 2] \rightarrow \mathbb{R}, \quad t \mapsto t^2 + t$$

Verify that the curve is simple, differentiable, and regular. Compute the curve integral $\int_u f \, dl$, where

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \rightarrow 3x^3$$

is the scalar field.

Solution 1 This is just a fancy way of describing integration by substitution from Analysis I. We first verify that the curve is simple, differentiable, and regular.

- The curve is simple if it does not intersect itself. This means that $u(t_1) \neq u(t_2)$ for $t_1 \neq t_2$. In this case, the curve $u(t) = t^2 + t$ is a strictly increasing function on the interval $[1, 2]$. We conclude that the curve is simple.
- We see that the curve is differentiable because its only component is differentiable.
- The curve is regular because the derivative $\dot{u}(t) = 2t + 1$ is not zero over the interval $(1, 2)$.

The curve integral of f over Γ is given by

$$\int_u f \, dl = \int_1^2 f(u(t)) |\dot{u}(t)| \, dt = \int_1^2 3(u(t))^3 |2t + 1| \, dt = \int_1^2 3(t^2 + t)^3 |2t + 1| \, dt.$$

We have $|2t + 1| = 2t + 1$ over the interval $[1, 2]$. Hence

$$\int_u f \, dl = 3 \int_1^2 (t^2 + t)^3 (2t + 1) \, dt.$$

This integral can be computed using standard methods of integration. One straight-forward but technical solution is to just expand the polynomial that we integrate. However, a simpler method uses substitution:

$$\int_u f \, dl = \frac{3}{4} \int_1^2 4(t^2 + t)^3 (2t + 1) \, dt = \frac{3}{4} \int_1^2 \partial_t (t^2 + t)^4 \, dt = \frac{3}{4} (t^2 + t)^4 \Big|_{t=1}^{t=2} = \frac{3}{4} (6^4 - 2^4). \quad (1)$$

This simplifies to

$$\int_u f \, dl = \frac{3}{4} \cdot 2^4 (3^4 - 1) = \frac{3}{4} \cdot 16 \cdot 80 = 3 \cdot 320 = 960. \quad (2)$$

Exercise 2 (vector analysis in 1D) Let $\Omega \subseteq (a, b)$ be an open interval in one-dimensional space.

- Explain why there cannot be a simple closed continuous curve in Ω .
- When $\Omega = (-10, 10)$, compute the integral of the scalar field

$$f(x) = \frac{x}{\sqrt{1+x^2}}$$

along the curves

$$\begin{aligned}\gamma_1 &: [0, 1] \rightarrow \Omega, & t &\mapsto (2t - 1), \\ \gamma_2 &: [-1, 1] \rightarrow \Omega, & t &\mapsto (t), \\ \gamma_3 &: [0, 1] \rightarrow \Omega, & t &\mapsto (1 - 2t), \\ \gamma_4 &: [0, 1] \rightarrow \Omega, & t &\mapsto (-1 + 2t^5),\end{aligned}$$

Compute the tangent vectors $\dot{\gamma}(t)$.

- Compute the integral of the vector field

$$F(x) = \begin{pmatrix} xe^{x^2} \end{pmatrix} \quad (3)$$

along the curve γ_4 . Find a potential for this vector field, and write down the general form of all potentials.

Solution 2 This is content of Analysis 1 but repackaged in the manner of vector analysis.

- It is visually clear that any closed continuous curve in Ω would have to intersect itself. Formally, one can use the intermediate value theorem.
- Before we compute all the line integrals, we note that the curves γ_1, γ_3 , and γ_4 map the interval $[0, 1]$ to the interval $[-1, 1]$. Hence, we can solve the corresponding line integrals by a change of variables in the line integral for γ_2 . Furthermore, we note that γ_3 is the reparameterization of γ_1 in the opposite direction. Hence, we conclude from Exercise 5 on Exercise Sheet 4, that the corresponding line integrals must be equal. Let us now start the computations with γ_2 :

$$\int_{\gamma_2} f \, dl = \int_{-1}^1 f(t) \, dt = \int_{-1}^1 \frac{t}{\sqrt{1+t^2}} \, dt = 0, \quad (4)$$

since the integrand is odd around $t = 0$. For γ_1 , we have

$$\int_{\gamma_1} f \, dl = \int_0^1 f(\gamma_1(t)) |\dot{\gamma}_1(t)| \, dt = 2 \int_0^1 f(2t - 1) \, dt = \int_{-1}^1 f(s) \, ds = 0 = \int_{\gamma_2} f \, dl = 0. \quad (5)$$

As already pointed out previously, we must have $\int_{\gamma_3} f \, dl = \int_{\gamma_1} f \, dl = 0$. Finally, we find for γ_4 that $\dot{\gamma}_4(t) = 10t^4 \geq 0$ for $t \in [0, 1]$ and thus

$$\int_{\gamma_4} f \, dl = \int_0^1 f(\gamma_4(t)) \dot{\gamma}_4(t) \, dt = \int_{-1}^1 f(u) \, du = \int_{\gamma_2} f \, dl = 0, \quad (6)$$

where we have used the substitution $u = \gamma_4(t) = -1 + 2t^5$ with $du = \dot{\gamma}_4(t) \, dt$.

- The general form of a potential for F is $f(x) = \frac{1}{2}e^{x^2} + C$, where C is an arbitrary constant. We set $C = -1/2$ such that $f(0) = 0$. For the curve integral of F along γ_4 , we thus find that

$$\int_{\gamma_4} F \, dl = f(\gamma_4(1)) - f(\gamma_4(0)) = f(1) - f(-1) = \frac{1}{2}(e - e^{(-1)^2}) = 0. \quad (7)$$

Exercise 3 We review notions of potentials and conservative vector fields. Let $\Omega \subseteq \mathbb{R}^n$ be open. Suppose we have a vector field $F = (F_1, \dots, F_n) \in C^1(\Omega, \mathbb{R}^n)$. Recall that we have introduced the condition

$$\partial_i F_j = \partial_j F_i, \quad 1 \leq i, j \leq n. \quad (8)$$

- Suppose that $n = 2$. Show that F satisfies (8) if and only if it is curl-free: $\text{curl } F = 0$.
- Suppose that $n = 3$. Show that F satisfies (8) if and only if it is curl-free: $\text{curl } F = 0$.
- Suppose that $n = 1$. Show that F satisfies (8).
- Suppose that F admits a potential $f \in C^1(\Omega, \mathbb{R})$, so that $\nabla f = F$. Show that if $\gamma : [a, b] \rightarrow \Omega$ is a simple regular curve, then

$$\int_{\gamma} F \, dl = f(\gamma(b)) - f(\gamma(a)). \quad (9)$$

Show that if γ is closed, then

$$\int_{\gamma} F \, dl = 0. \quad (10)$$

Solution 3 • We recall that the curl of a vector field $F = (F_1, F_2)$ is given by

$$\text{curl } F = (\partial_1 F_2 - \partial_2 F_1). \quad (11)$$

Hence, it is clear that $\text{curl } F = 0$ if and only if F satisfies (8).

- We recall that the curl of a vector field $F = (F_1, F_2, F_3)$ is given by

$$\operatorname{curl} F = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}. \quad (12)$$

Hence, it is clear that $\operatorname{curl} F = 0$ if and only if F satisfies (8). Also, recall Exercise 6 from Exercise Sheet 3. On the one hand, we noticed there that the curl of a vector field is given by the off-diagonal entries of the anti-symmetric part of the Jacobian matrix. On the other hand, condition (8) is equivalent to the statement that the Jacobian matrix of F is symmetric. We therefore conclude that the curl vanishes if and only if the Jacobian matrix is symmetric.

- For $n = 1$, we have $F = (F_1)$ and the condition (8) is trivially satisfied.
- We have $\nabla f = F$. Hence, we can write the line integral as

$$\int_{\gamma} F \, dl = \int_a^b F(\gamma(t)) \cdot \dot{\gamma}(t) \, dt \quad (13)$$

$$= \int_a^b \nabla f(\gamma(t)) \cdot \dot{\gamma}(t) \, dt = \int_a^b \frac{d}{dt} f(\gamma(t)) \, dt = f(\gamma(b)) - f(\gamma(a)). \quad (14)$$

If γ is closed, then $\gamma(a) = \gamma(b)$ and we find that the line integral vanishes.

Exercise 4 We introduce the following curves:

$$\begin{aligned} \gamma : [0, 1] &\rightarrow \mathbb{R}^3, & t &\mapsto (3, t^2, 4t), \\ \delta : [1, \infty) &\rightarrow \mathbb{R}^2, & t &\mapsto (5, e^{-t}) \end{aligned}$$

For each curve

- compute the tangent vector
- compute the speed of the curve
- find the unit tangent vector
- for δ , find the unit normal along the curve that is the 90 degree clockwise rotation of unit tangent
- argue why it is a regular curve
- and compute the length of the curve.

Solution 4 • We compute the tangent vectors:

$$\dot{\gamma}(t) = \begin{pmatrix} 0 \\ 2t \\ 4 \end{pmatrix}, \quad \dot{\delta}(t) = \begin{pmatrix} 0 \\ -e^{-t} \end{pmatrix}. \quad (15)$$

- The speed of the curve is given by the norm of the tangent vector:

$$|\dot{\gamma}(t)| = \sqrt{4t^2 + 16} = 2\sqrt{t^2 + 4}, \quad |\dot{\delta}(t)| = e^{-t}. \quad (16)$$

- The unit tangent vector is given by

$$\hat{t}_{\gamma}(t) = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} = \frac{1}{\sqrt{t^2 + 4}} \begin{pmatrix} 0 \\ t \\ 2 \end{pmatrix}, \quad (17)$$

$$\hat{t}_{\delta}(t) = \frac{\dot{\delta}(t)}{|\dot{\delta}(t)|} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \quad (18)$$

Note that the unit tangent vector of δ is constant!

- The unit normal vector is given by the 90 degree clockwise rotation of the unit tangent vector. The corresponding rotation matrix is

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (19)$$

Hence, we find

$$\hat{n}_{\delta}(t) = R\hat{t}_{\delta}(t) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \quad (20)$$

- The curves are regular because their tangent vectors never vanish.
- The length of the curves is given by the integral of the speed: For γ , we find

$$\int_0^1 |\dot{\gamma}(t)| dt = 2 \int_0^1 \sqrt{t^2 + 4} dt = 4 \int_0^1 \sqrt{(t/2)^2 + 1} dt \quad (21)$$

$$= 8 \int_0^{1/2} \sqrt{u^2 + 1} du = 4(\sqrt{2} + \sinh^{-1}(1/2)), \quad (22)$$

$$(23)$$

where we have used the substitution $u = t/2$ with $du = dt/2$. The last integral can be found with trigonometric substitution.

For δ , we find

$$\int_1^{\infty} |\dot{\delta}(t)| dt = \int_1^{\infty} e^{-t} dt = e^{-1}. \quad (24)$$

Note that the curve is parametrized for $t \in [1, \infty)$, but still has a finite length!

Exercise 5 We consider the vector field

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \mapsto (x^3, y^3)$$

We want to find a potential over the domain $\Omega = \mathbb{R}^2$. Fix a constant of integration at $(0, 0)$ and define a potential via the integral of the vector field F along a simple regular curve going from $(0, 0)$ to (x, y) .

Solution 5 The most simple among the simple regular curves from $(0, 0)$ to (x, y) is the straight line:

$$\gamma : [0, 1] \rightarrow \mathbb{R}^2, \quad t \mapsto (tx, ty).$$

We compute that $\dot{\gamma}(t) = (x, y)$. We fix some constant of integration $f(0, 0) = C$ for our yet-to-be-found potential $f \in C^1(\mathbb{R}^2, \mathbb{R})$. For any $(x, y) \in \mathbb{R}^2$, we now compute

$$f(x, y) - f(0, 0) = \int_0^1 (t^3 x^3, t^3 y^3) \cdot (x, y) \, dt = (x^4 + y^4) \int_0^1 t^3 \, dt = \frac{1}{4} (x^4 + y^4). \quad (25)$$

Therefore,

$$f(x, y) = \frac{1}{4} (x^4 + y^4) + C. \quad (26)$$

Indeed, one easily verifies that $\nabla f = F$.

Exercise 6 The closed curve

$$\gamma(t) = (\sin(t)(1 + 0.5 \sin(2t)), \cos(t)(1 + 0.5 \sin(2t)))$$

encircles a domain Ω in counterclockwise direction. Find the tangent $\dot{\gamma}(t)$, the unit tangent $\tau(t)$ and the outward pointing unit normal $\vec{n}(t)$. Only simplify as much as reasonable.

Solution 6 We calculate:

$$\dot{\gamma}(t) = (\cos(t)(1 + 0.5 \sin(2t)) + \sin(t) \cos(2t), -\sin(t)(1 + 0.5 \sin(2t)) + \cos(t) \cos(2t))$$

With that:

$$\begin{aligned} |\dot{\gamma}(t)|^2 &= \cos(t)^2(1 + 0.5 \sin(2t))^2 + \sin(t)^2 \cos(2t)^2 + 2 \cos(t)(1 + 0.5 \sin(2t)) \sin(t) \cos(2t) \\ &\quad + \sin(t)^2(1 + 0.5 \sin(2t))^2 + \cos(t)^2 \cos(2t)^2 - 2 \sin(t)(1 + 0.5 \sin(2t)) \cos(t) \cos(2t) \\ &= (1 + 0.5 \sin(2t))^2 + \cos(2t)^2 + 2 \cos(t)(1 + 0.5 \sin(2t)) \sin(t) \cos(2t) \\ &\quad - 2 \sin(t)(1 + 0.5 \sin(2t)) \cos(t) \cos(2t) \\ &= (1 + 0.5 \sin(2t))^2 + \cos(2t)^2. \end{aligned}$$

Hence

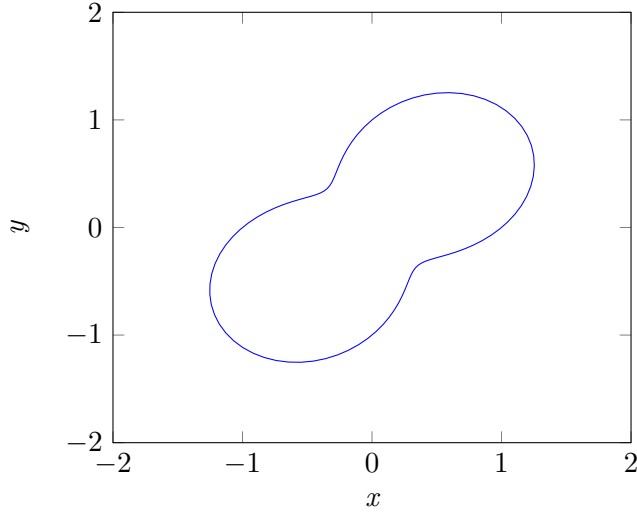
$$|\dot{\gamma}(t)| = \sqrt{(1 + 0.5 \sin(2t))^2 + \cos(2t)^2}.$$

We want to compute the tangent vector:

$$\tau(t) = \left(\frac{\cos(t)(1 + 0.5 \sin(2t)) + \sin(t) \cos(2t)}{\sqrt{(1 + 0.5 \sin(2t))^2 + \cos(2t)^2}}, \frac{-\sin(t)(1 + 0.5 \sin(2t)) + \cos(t) \cos(2t)}{\sqrt{(1 + 0.5 \sin(2t))^2 + \cos(2t)^2}} \right)$$

Accordingly, the normal vector is:

$$\vec{n}(t) = \left(\frac{-\sin(t)(1 + 0.5 \sin(2t)) + \cos(t) \cos(2t)}{\sqrt{(1 + 0.5 \sin(2t))^2 + \cos(2t)^2}}, -\frac{\cos(t)(1 + 0.5 \sin(2t)) + \sin(t) \cos(2t)}{\sqrt{(1 + 0.5 \sin(2t))^2 + \cos(2t)^2}} \right)$$



Exercise 7 We work over the quadratic domain

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 < x_1 < 1, -1 < x_2 < 1\}.$$

Compute the integral $\iint_{\Omega} \operatorname{div} \vec{F} \, dx_1 dx_2$, where

$$\vec{F}(x_1, x_2) = \left(\sin(x_1)x_2, (x_1^2 + x_2)^5 \right)$$

Solution 7 We make use of the Divergence theorem to express the volume integral as a curve integral.

$$\int \int_{\Omega} \nabla \cdot \vec{F} \, dx_1 \, dx_2 = \int_{\partial \Omega} \vec{F} \cdot \vec{n} \, d\ell$$

$$\begin{aligned}
&= \int_{-1}^1 \vec{F}(x_1, -1) \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dx_1 + \int_{-1}^1 \vec{F}(-x_1, 1) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dx_1 \\
&\quad + \int_{-1}^1 \vec{F}(-1, -x_2) \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} dx_2 + \int_{-1}^1 \vec{F}(1, x_2) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dx_2 \\
&= - \int_{-1}^1 (x_1^2 - 1)^5 dx_1 + \int_{-1}^1 (x_1^2 + 1)^5 dx_1 \\
&\quad + (-1) \int_{-1}^1 \sin(-1)(-x_2) dx_2 + \int_{-1}^1 \sin(1)x_2 dx_2 \\
&= \int_{-1}^1 (1 - x_1^2)^5 dx_1 + \int_{-1}^1 (1 + x_1^2)^5 dx_1 \\
&\quad + (-1) \int_{-1}^1 \sin(-1)(-x_2) dx_2 + \int_{-1}^1 \sin(1)x_2 dx_2.
\end{aligned}$$

We observe the last two integrals are zero: either because x_2 is an odd function, or because

$$(-1) \int_{-1}^1 \sin(-1)(-x_2) dx_2 + \int_{-1}^1 \sin(1)x_2 dx_2 = \int_{-1}^1 \sin(1)(-x_2) dx_2 + \int_{-1}^1 \sin(1)x_2 dx_2 = 0.$$

To evaluate the remaining two integrals, we can either compute the quintic powers manually, which is a lot of work, or we utilize the binomial theorem:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

With that, we can simplify the calculation as follows:

$$\begin{aligned}
\int \int_{\Omega} \nabla \cdot \vec{F} dx_1 dx_2 &= \int_{\partial\Omega} \vec{F} \cdot \vec{n} d\ell, \\
&= \int_{-1}^1 (1 - x_1^2)^5 dx_1 + \int_{-1}^1 (1 + x_1^2)^5 dx_1 \\
&= \sum_{k=0}^5 \binom{5}{k} (-1)^k \int_{-1}^1 (x)^{2k} dx_1 + \sum_{k=0}^5 \binom{5}{k} \int_{-1}^1 (x)^{2k} dx_1 \\
&= \sum_{k=0}^5 \binom{5}{k} (-1)^k \left[\frac{x^{2k+1}}{2k+1} \right]_{-1}^1 + \sum_{k=0}^5 \binom{5}{k} \left[\frac{x^{2k+1}}{2k+1} \right]_{-1}^1 \\
&= \sum_{k=0}^5 \binom{5}{k} \left((-1)^k + 1 \right) \left[\frac{x^{2k+1}}{2k+1} \right]_{-1}^1
\end{aligned}$$

For $k = 1, 3, 5$, we get $((-1)^k + 1) = 0$. So we only need the terms with $k = 0, 2, 4$. Thus

$$\sum_{k=0}^5 \binom{5}{k} \left((-1)^k + 1 \right) \left[\frac{x^{2k+1}}{2k+1} \right]_{-1}^1$$

$$\begin{aligned}
&= 2 \binom{5}{0} \left[\frac{x^{0+1}}{0+1} \right]_{-1}^1 + 2 \binom{5}{2} \left[\frac{x^{4+1}}{4+1} \right]_{-1}^1 + 2 \binom{5}{4} \left[\frac{x^{8+1}}{8+1} \right]_{-1}^1 \\
&= 2 \binom{5}{0} \left[\frac{x^1}{1} \right]_{-1}^1 + 2 \binom{5}{2} \left[\frac{x^5}{5} \right]_{-1}^1 + 2 \binom{5}{4} \left[\frac{x^9}{9} \right]_{-1}^1 \\
&= 4 \binom{5}{0} + \frac{4}{5} \binom{5}{2} + \frac{4}{9} \binom{5}{4} \\
&= \frac{128}{9}.
\end{aligned}$$