Analysis III - 203(d)

Winter Semester 2024

Session 2: September 19, 2024

Exercise 1 *Sketch the level sets of the functions*

$$f(x_1, x_2) = x_1 + x_2, \quad g(x_1, x_2) = x_1^2 x_2$$

for the values 0, 1, *and* -1. *Compute the gradients* ∇f *and* ∇g.

Solution 1

$$\nabla f(x_1, x_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \nabla g(x_1, x_2) = \begin{pmatrix} 2x_1 x_2 \\ x_1^2 \end{pmatrix}$$

For the level sets of $f(x_1, x_2)$ *consider:*

$$x_1 + x_2 = C \implies x_2 = -x_1 + C$$

therefore the level sets are lines that intersect the axis x_2 *at* C *with slope* -1. *One can use a similar approach for* $g(x_1, x_2)$. *Note that for* $C = 0$*:*

$$x_1^2 x_2 = 0 \implies x_1 = 0 \text{ or } x_2 = 0$$

which gives that the level sets for $C = 0$ *are the two axis* x_1 *and* x_2.

Exercise 2 *Compute the curl and the divergence of the following vector field:*

$$\vec{f}(x_1, x_2, x_3) = \left(\ x_1 x_2,\ x_3^2 x_1,\ e^{x_1 - x_2}\ \right)$$

Solution 2

$$\nabla \times \vec{f}(x_1, x_2, x_3) = \left(\ -e^{x_1 - x_2} - 2x_1 x_3,\ -e^{x_1 - x_2},\ x_3^2 - x_1\ \right)$$
$$\nabla \cdot \vec{f}(x_1, x_2, x_3) = x_2$$

Exercise 3 *Show that for every three-dimensional vector field* $\vec{f} \colon \mathbb{R}^3 \to \mathbb{R}^3$ *we have*

$$\nabla \cdot \left(\nabla \times \vec{f}\right) = 0$$

Show that for every scalar field $g \colon \mathbb{R}^3 \to \mathbb{R}$ *we have*

$$\nabla \times (\nabla g) = 0$$

Consider the following vector field:

$$\vec{A}(x_1, x_2, x_3) := (\ \sin(x_2),\ \sin(x_3),\ \sin(x_1)\ )$$

Is this vector field a gradient of some scalar field?

Solution 3 *Let $\vec{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$ then:*

$$\begin{aligned}\nabla \cdot (\nabla \times \vec{f}(x_1, x_2, x_3)) &= (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \cdot (\partial_{x_2} f_3 - \partial_{x_3} f_2, \partial_{x_3} f_1 - \partial_{x_1} f_3, \partial_{x_1} f_2 - \partial_{x_2} f_1) \\ &= \partial_{x_1 x_2} f_3 - \partial_{x_1 x_3} f_2 + \partial_{x_2 x_3} f_1 - \partial_{x_2 x_1} f_3 + \partial_{x_3 x_1} f_2 - \partial_{x_3 x_2} f_1\end{aligned}$$

We can interchange the order of the partial derivatives, since $\vec{f} \in \mathrm{C}^2$ to arrive at:

$$\begin{aligned}\nabla \cdot (\nabla \times \vec{f}(x_1, x_2, x_3)) &= \partial_{x_1 x_2} f_3 - \partial_{x_2 x_1} f_3 + \partial_{x_3 x_1} f_2 - \partial_{x_1 x_3} f_2 + \partial_{x_2 x_3} f_1 - \partial_{x_3 x_2} f_1 \\ &= \partial_{x_2 x_1} f_3 - \partial_{x_2 x_1} f_3 + \partial_{x_1 x_3} f_2 - \partial_{x_1 x_3} f_2 + \partial_{x_3 x_2} f_1 - \partial_{x_3 x_2} f_1 = 0 \\ &\implies \nabla \cdot (\nabla \times \vec{f}(x_1, x_2, x_3)) = 0\end{aligned}$$

Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ then:

$$\nabla \times (\nabla g(x_1, x_2, x_3)) = \nabla \times \begin{pmatrix} \partial_{x_1} g \\ \partial_{x_2} g \\ \partial_{x_3} g \end{pmatrix} = \begin{pmatrix} \partial_{x_2 x_3} g - \partial_{x_3 x_2} g \\ \partial_{x_3 x_1} g - \partial_{x_1 x_3} g \\ \partial_{x_1 x_2} g - \partial_{x_2 x_1} g \end{pmatrix} = \begin{pmatrix} \partial_{x_2 x_3} g - \partial_{x_2 x_3} g \\ \partial_{x_1 x_3} g - \partial_{x_1 x_3} g \\ \partial_{x_1 x_2} g - \partial_{x_1 x_2} g \end{pmatrix} = \vec{0}$$

$$\implies \nabla \times (\nabla g(x_1, x_2, x_3)) = 0$$

$$\vec{\nabla} \times \vec{A}(x_1, x_2, x_3) := (\, -\cos(x_3), \cos(x_1), -\cos(x_2)\,)$$

The curl of this vector field is not zero. If it were a gradient, then it would be zero. So it cannot be a gradient of any function.

Exercise 4 *Given scalar fields $f, g : \mathbb{R}^3 \to \mathbb{R}$ and vector fields $\vec{A}, \vec{B} : \mathbb{R}^3 \to \mathbb{R}^3$, show that*

$$\begin{gathered}\nabla\,(f \cdot g) = f \cdot \nabla g + g \cdot \nabla f, \\ \nabla \cdot \left(f \cdot \vec{A}\right) = (\nabla f) \cdot \vec{A} + f(\nabla \cdot \vec{A}), \\ \nabla \cdot \left(\vec{A} \times \vec{B}\right) = \left(\nabla \times \vec{A}\right) \cdot \vec{B} - \vec{A} \cdot \left(\nabla \times \vec{B}\right), \\ \Delta(fg) = f\Delta g + 2\nabla(f) \cdot \nabla(g) + g\Delta f.\end{gathered}$$

Suppose that we also have two real numbers $\alpha, \beta \in \mathbb{R}$. Show that

$$\begin{gathered}\nabla(\alpha f + \beta g) = \alpha \nabla f + \beta \nabla g, \\ \nabla \cdot (\alpha \vec{A} + \beta \vec{B}) = \alpha \nabla \cdot \vec{A} + \beta \nabla \cdot \vec{B}, \\ \nabla \times (\alpha \vec{A} + \beta \vec{B}) = \alpha \nabla \times \vec{A} + \beta \nabla \times \vec{B}.\end{gathered}$$

Solution 4 *For the first identity, we use the product rule:*

$$\begin{aligned}\nabla\,(f \cdot g) &= (\partial_1 (f \cdot g), \cdots, \partial_n (f \cdot g)) \\ &= (\partial_1 f \cdot g + f \cdot \partial_1 g, \cdots, \partial_n f \cdot g + f \cdot \partial_n g)\end{aligned}$$

$$
\begin{aligned}
&= (\partial_1 f \cdot g, \cdots, \partial_n f \cdot g) + (f \cdot \partial_1 g, \cdots, f \cdot \partial_n g) \\
&= g\,(\partial_1 f, \cdots, \partial_n f) + f\,(\partial_1 g, \cdots, \partial_n g) \\
&= f \cdot \nabla g + g \cdot \nabla f.
\end{aligned}
$$

The second identity, again, can be proved using the product rule:

$$
\begin{aligned}
\nabla \cdot \left(f \cdot \vec{A}\right) &= \partial_1(f \cdot A_1) + \cdots + \partial_n(f \cdot A_n) \\
&= \partial_1(f \cdot A_1) + \cdots + \partial_n(f \cdot A_n) \\
&= (\partial_1 f) \cdot A_1) + \cdots + (\partial_n f) \cdot A_n + f \cdot \partial_1(A_1) + \cdots + f \cdot \partial_n(A_n) \\
&= \nabla(f) \cdot \vec{A} + f \cdot \nabla \cdot \vec{A},
\end{aligned}
$$

As for the third identity:

$$
\begin{aligned}
\nabla \cdot \left(\vec{A} \times \vec{B}\right) &= \nabla \cdot (A_2B_3 - A_3B_2, -A_1B_3 + A_3B_1, A_1B_2 - A_2B_1) \\
&= \partial_1\left(A_2B_3 - A_3B_2\right) + \partial_2\left(-A_1B_3 + A_3B_1\right) + \partial_3\left(A_1B_2 - A_2B_1\right) \\
&= +\partial_1\left(A_2B_3\right) - \partial_1\left(A_3B_2\right) \\
&\quad - \partial_2\left(A_1B_3\right) + \partial_2\left(A_3B_1\right) \\
&\quad + \partial_3\left(A_1B_2\right) - \partial_3\left(A_1B_2\right) \\
&= +\partial_1\left(A_2\right)B_3 + A_2\partial_1\left(B_3\right) - \partial_1\left(A_3\right)B_2 - A_3\partial_1\left(B_2\right) \\
&\quad - \partial_2\left(A_1\right)B_3 - A_1\partial_2\left(B_3\right) + \partial_2\left(A_3\right)B_1 + A_3\partial_2\left(B_1\right) \\
&\quad + \partial_3\left(A_1\right)B_2 + A_1\partial_3\left(B_2\right) - \partial_3\left(A_2\right)B_1 - A_2\partial_3\left(B_1\right) \\
&= +\partial_2\left(A_3\right)B_1 - \partial_3\left(A_2\right)B_1 - \partial_1\left(A_3\right)B_2 + \partial_3\left(A_1\right)B_2 + \partial_1\left(A_2\right)B_3 - \partial_2\left(A_1\right)B_3 \\
&\quad + A_1\partial_3\left(B_2\right) - A_1\partial_2\left(B_3\right) - A_2\partial_3\left(B_1\right) + A_2\partial_1\left(B_3\right) + A_3\partial_2\left(B_1\right) - A_3\partial_1\left(B_2\right) \\
&= (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A}.
\end{aligned}
$$

We can prove the fourth identity again via direct computation, which is perfectly valid. We can also use some of the identities already shown above to reduce the effort:

$$
\begin{aligned}
\Delta(fg) &= \nabla \cdot (\nabla(fg)) \\
&= \nabla \cdot (g\nabla(f) + f\nabla(g)) \\
&= \partial_1(g\nabla(f)_1 + f\nabla(g)_1) + \cdots + \partial_n(g\nabla(f)_n + f\nabla(g)_n) \\
&= \partial_1(g\nabla(f)_1) + \partial_1(f\nabla(g)_1) + \cdots + \partial_n(g\nabla(f)_n) + \partial_n(f\nabla(g)_n) \\
&= \partial_1(g\nabla(f)_1) + \cdots + \partial_n(g\nabla(f)_n) + \partial_1(f\nabla(g)_1) + \cdots + \partial_n(f\nabla(g)_n) \\
&= \nabla \cdot (g\nabla(f)) + \nabla \cdot (f\nabla(g)) \\
&= \nabla(g) \cdot \nabla(f) + g \cdot \nabla \cdot (\nabla f) + \nabla(f) \cdot \nabla(g) + f \cdot \nabla \cdot (\nabla g) = 2\nabla(f) \cdot \nabla(g) + f\Delta g + g\Delta f.
\end{aligned}
$$

The last three identities describe that gradient, divergence and curl are linear*, which is a concept that you have already seen in linear algebra. We verify by direct computation:*

$$
\nabla(\alpha f + \beta g) = (\alpha\partial_1 f_1 + \beta\partial_1 g_1, \cdots, \alpha\partial_n f_n + \beta\partial_n g_n)
$$

$$= (\alpha\partial_1 f_1, \cdots, \alpha\partial_n f_n) + (\beta\partial_1 g_1, \cdots, \beta\partial_n g_n)$$
$$= \alpha (\partial_1 f_1, \cdots, \partial_n f_n) + \beta (\partial_1 g_1, \cdots, \partial_n g_n).$$

Similarly,

$$\begin{aligned}
\nabla \cdot (\alpha\vec{A} + \beta\vec{B}) &= \nabla \cdot (\alpha A_1 + \beta B_1, \cdots, \alpha A_n + \beta B_n) \\
&= \alpha\partial_1 A_1 + \beta\partial_1 B_1 + \cdots + \alpha\partial_n A_n + \beta\partial_n B_n \\
&= \alpha\partial_1 A_1 + \cdots + \alpha\partial_n A_n + \beta\partial_1 B_1 + \cdots + \beta\partial_n B_n \\
&= \alpha (\partial_1 A_1 + \cdots + \partial_n A_n) + \beta (\partial_1 B_1 + \cdots + \partial_n B_n) \\
&= \alpha\nabla \cdot \vec{A} + \beta\nabla\vec{B}.
\end{aligned}$$

Lastly,

$$\begin{aligned}
&\nabla \times (\alpha\vec{A} + \beta\vec{B}) \\
&= \nabla \times (\alpha A_1 + \beta B_1, \alpha A_2 + \beta B_2, \alpha A_3 + \beta B_3) \\
&= (\alpha\partial_2 A_3 + \beta\partial_2 B_3 - \alpha\partial_3 A_2 - \beta\partial_3 B_2, -\alpha\partial_1 A_3 - \beta\partial_1 B_3 + \alpha\partial_3 A_1 + \beta\partial_3 B_1, \alpha\partial_1 A_2 + \beta\partial_1 B_2 - \alpha\partial_2 A_1 - \beta\partial_2 B_1) \\
&= (\alpha\partial_2 A_3 - \alpha\partial_3 A_2 + \beta\partial_2 B_3 - \beta\partial_3 B_2, -\alpha\partial_1 A_3 + \alpha\partial_3 A_1 - \beta\partial_1 B_3 + \beta\partial_3 B_1, \alpha\partial_1 A_2 - \alpha\partial_2 A_1 + \beta\partial_1 B_2 - \beta\partial_2 B_1) \\
&= \alpha (\partial_2 A_3 - \partial_3 A_2, -\partial_1 A_3 + \partial_3 A_1, \partial_1 A_2 - \partial_2 A_1) + \beta (\partial_2 B_3 - \partial_3 B_2, -\partial_1 B_3 + \partial_3 B_1, \partial_1 B_2 - \partial_2 B_1) \\
&= \alpha\nabla \times \vec{A} + \beta\nabla \times \vec{B}.
\end{aligned}$$

Exercise 5 *Consider the scalar field*

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \|x\|^{-1}$$

and compute its gradient and Laplacian. Show that Φ is harmonic if $n = 3$. When $n = 3$, then this is the graviational potential, and its gradient is the gravitational field.

Solution 5 *We remember that*

$$\nabla\|x\| = \frac{x}{\|x\|}.$$

Hence

$$\partial_i \Phi(x) = (-1)\|x\|^{-2} \cdot \frac{x_i}{\|x\|} = -\frac{x_i}{\|x\|^3}.$$

That mans

$$\nabla\Phi = -\frac{x}{\|x\|^3}.$$

We compute also the second derivative in direction i:

$$\partial_i\partial_i\Phi(x) = -\frac{1}{\|x\|^3} - x_i \cdot (-3)\|x\|^{-4} \frac{x_i}{\|x\|} = -\frac{1}{\|x\|^3} + 3\frac{x_i^2}{\|x\|^5}.$$

We easily see that the Laplacian equals

$$\Delta\Phi(x) = \frac{n-3}{\|x\|^3}$$

In particular, the scalar field Φ *is "harmonic" if* $n = 3$*: We call a scalar field harmonic if its Laplacian equals zero.*