

Analysis III - 203(d)

Winter Semester 2024

Session 6: October 17, 2024

Exercise 1 We have the vector field

$$\vec{F}(x_1, x_2) = (x_1 x_2, x_2^2).$$

and the domains

$$\begin{aligned}\Omega_1 &:= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}, \\ \Omega_2 &:= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1, x_1 > 0\},\end{aligned}$$

Verify Green's theorem for the vector field $\vec{F}$ with the domains Ω_1 and Ω_2 . You need to find parameterizations of the boundary first.

Solution 1 Green's theorem over some domain Ω is given by

$$\int_{\partial\Omega} \vec{F} \cdot d\vec{s} = \int \int_{\Omega} \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} d\Omega.$$

Here, $\partial\Omega$ is a curve parameterized in counterclockwise direction.

We consider the domain Ω_1 . A counterclockwise parameterisation for the boundary is given by

$$\gamma : [0, 2\pi] \mapsto \mathbb{R}^2, \quad t \mapsto (\cos t, \sin t).$$

$$\begin{aligned}\int_{\partial\Omega} \vec{F} \cdot d\vec{s} &= \int_0^{2\pi} \begin{pmatrix} \cos t \sin t \\ \sin^2 t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt \\ &= \int_0^{2\pi} -\sin^2 t \cos t + \sin^2 t \cos t dt = 0\end{aligned}$$

On the other hand,

$$\int \int_{\Omega} \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} d\Omega = \int \int_{\Omega} -x_1 d\Omega.$$

There are different ways to compute the last integral. For example, since Ω is symmetric along the x_2 axis but $-x_1$ is obviously an odd function, we can conclude that

$$\int \int_{\Omega} -x_1 d\Omega = 0.$$

Alternatively, we can use radial coordinates for the disk:

$$\int \int_{\Omega} -x_1 \, d\Omega = \int_0^{2\pi} \int_0^1 -r \cos(\theta) r \, d\theta \, dr = - \int_0^{2\pi} \cos(\theta) \, d\theta \int_0^1 r^2 \, dr = 0,$$

where we have used that integrating the cosine function over a period is 0. Yet another possibility is

$$\int \int_{\Omega} -x_1 \, d\Omega = - \int_{-1}^1 \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} x_1 \, dx_2 \, dx_1 = \int_{-1}^1 2x_1 \sqrt{1-x_1^2} \, dx_1 = 0,$$

where we have used that the last function is odd. In any case, the integral of the domain and the integral along the boundary give the same value, thereby verifying Green's theorem.

We consider domain Ω_2 . A parameterisation for the boundary is given by two separate curves, one for the circular arc and one for the straight line. For the circular arc we have

$$\gamma : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \mapsto \mathbb{R}^2, \quad t \mapsto (\cos t, \sin t) \quad (1)$$

and for the vertical line we have

$$\delta : [-1, 1] \mapsto \mathbb{R}^2, \quad t \mapsto (0, -t). \quad (2)$$

With that, we compute

$$\begin{aligned} \int_{\partial\Omega} \vec{F} \cdot d\vec{s} &= \int_{\gamma} \vec{F} \cdot d\vec{s} + \int_{\delta} \vec{F} \cdot d\vec{s} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \begin{pmatrix} \sin t \cos t \\ \sin^2 t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt + \int_{-1}^1 \begin{pmatrix} 0 \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &= \int_0^{2\pi} -\sin^2 t \cos t + \sin^2 t \cos t \, dt - \int_{-1}^1 t^2 \, dt \\ &= - \left[\frac{1}{3} t^3 \right]_{-1}^1 = -\frac{2}{3}. \end{aligned}$$

We compute the integral over the domain:

$$\begin{aligned} \int \int_{\Omega} \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \, d\Omega &= \int \int_{\Omega} -x_1 \, d\Omega \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 -r \cos(\theta) r \, d\theta \, dr \\ &= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta) \, d\theta \int_0^1 r^2 \, dr = [\sin \theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-\frac{1}{3} r^3 \right]_0^1 = -\frac{2}{3}. \end{aligned}$$

Both integrals give the same value, thereby verifying Green's theorem.

Exercise 2 Consider the triangle domain

$$T := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0, x_1 + x_2 < 1\}.$$

and the vector field

$$\vec{F}(x_1, x_2) = \left(x_1 x_2 + \frac{x_1}{1 + x_1^2 + x_2^2}, x_1 x_2 + \frac{x_2}{1 + x_1^2 + x_2^2} \right)$$

Find the curve integral of $\vec{F}$ along the boundary of T using Green's theorem.

Solution 2

$$\begin{aligned} \int_{\partial\Omega} \vec{F} \cdot d\vec{s} &= \int \int_{\Omega} \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} d\Omega \\ &= \int \int_{\Omega} x_2 + \frac{-2x_1 x_2}{(1 + x_1^2 + x_2^2)^2} - x_1 - \frac{-2x_1 x_2}{(1 + x_1^2 + x_2^2)^2} d\Omega \\ &= \int \int_{\Omega} x_2 - x_1 d\Omega \\ &= \int_0^1 \int_0^{1-x_1} x_2 - x_1 dx_2 dx_1 \\ &= \int_0^1 \frac{1}{2} (1 - x_1)^2 - x_1 (1 - x_1) dx \\ &= \left[-\frac{1}{6} (1 - x_1)^3 - \frac{1}{2} x_1^2 + \frac{1}{3} x_1^3 \right]_0^1 = -\frac{1}{2} + \frac{1}{3} - \frac{-1}{6} = 0 \end{aligned}$$

Exercise 3 Consider the parabolic arc

$$\Gamma := \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 < x_1 < 1, x_2 = 3(1 - x_1^2)\}.$$

Find the curve integral $\int_{\Gamma} \vec{F} \cdot \vec{n} dl$, where

$$\vec{F}(x_1, x_2) = (x_1(2 - \cos(x_1 x_2))^2, x_2(2 + \cos(x_1 x_2))^2)$$

and where $\vec{n}$ is the unit vector along Γ , perpendicular to Γ and having non-negative x_2 component.

Solution 3 The Idea is to make use of divergence theorem. We do this by closing the curve γ by introducing the curve

$$\delta : [-1, 1] \mapsto \mathbb{R}^2, \quad t \mapsto (t, 0).$$

This allows us to reformulate the problem as follows:

$$\int_{\Gamma} \vec{F} \cdot \vec{n} dl = \int_{\Gamma \cup \Delta} \vec{F} \cdot \vec{n} dl - \int_{\Delta} \vec{F} \cdot \vec{n} dl$$

For the first integral on the right-hand side we use the divergence theorem

$$\int_{\Gamma \cup \Delta} \vec{F} \cdot \vec{n} \, d\ell = \int_{\Omega} \nabla \cdot \vec{F} \, d\Omega = \int_{\Omega} 4 \, d\Omega.$$

We need to compute the area of the domain Ω and proceed as follows:

$$\begin{aligned} 4 \int_{\Omega} 1 \, d\Omega &= 4 \int_{-1}^1 \int_0^{3(1-x_1^2)} 1 \, dx_2 \, dx_1 \\ &= 4 \int_{-1}^1 [x_2]_0^{3(1-x_1^2)} \, dx_1 \\ &= 4 \int_{-1}^1 3(1-x_1^2) \, dx_1 \\ &= 4 [3x_1 - x_1^3]_{-1}^1 = 4 ([3 - 1] - [-3 + 1]) = 16. \end{aligned}$$

The value of the second integral is given by the integral of $\vec{F}$ against the vector $(0, -1)$:

$$\int_{\Delta} \vec{F} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \, dl = \int_{-1}^1 0 \, dx_1 = 0.$$

We conclude that:

$$\int_{\Gamma} \vec{F} \cdot \vec{n} \, dl = \int_{\Gamma \cup \Delta} \vec{F} \cdot \vec{n} \, dl = [12x_1 - 4x_1^3]_{-1}^1 = (12 - 4) - (-12 + 4) = 16.$$

Exercise 4 Find the tangential vector $\dot{\gamma}(t)$, the unit tangential vector $\vec{\tau}$ and the unit normal $\vec{n}$ of the simple closed curve

$$\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2, \quad t \mapsto (\cos(t), \sin(t)(1 + \sin(2t)^2)).$$

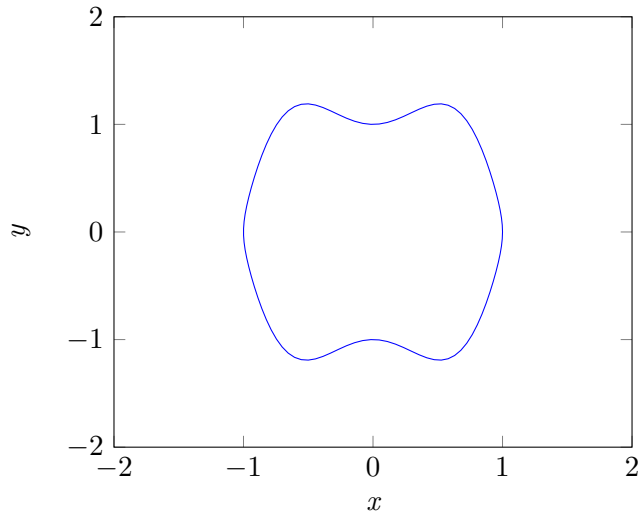
Find the values of γ and $\vec{\tau}$ for a few values of $t \in [0, 2\pi]$, such as $t = \frac{\pi}{4}, 2\frac{\pi}{4}, 3\frac{\pi}{4}, \dots, 7\frac{\pi}{4}$

Solution 4 Some standard calculations show that

$$\begin{aligned} \dot{\gamma}(t) &= (\sin(t), \cos(t) + \cos(t) \sin(2t)^2 + \sin(t) 2 \sin(2t) \cos(2t) 2) \\ &= (\sin(t), \cos(t) + \cos(t) \sin(2t)^2 + 4 \sin(t) \sin(2t) \cos(2t)) \end{aligned}$$

Hence,

$$\begin{aligned} |\dot{\gamma}(t)| &= \sqrt{\sin(t)^2 + \cos(t)^2 + \cos(t)^2 \sin(2t)^4 + 16 \sin(t)^2 \sin(2t)^2 \cos(2t)^2 + 2 \cos(t) \sin(2t)^2 + 2 \cos(t) 4 \sin(t) \sin(2t) \cos(2t) + 2 \sin(2t)^2 4 \sin(t) \sin(2t) \cos(2t)} \\ &= \sqrt{\frac{1}{16} (6 \cos(t) + 3 \cos(3t) - 5 \cos(5t))^2 + \sin(t)^2} \end{aligned}$$



Exercise 5 Find the area of the graph of ϕ over $\Omega = [0, 1] \times [0, 1]$, where

$$\phi(s, t) := \sqrt{s^2 + t^2}$$

Find the integral of the function

$$f(x_1, x_2, x_3) := x_1 x_2 x_3$$

over the graph of ϕ over Ω .

Solution 5 We compute the partial derivatives of the function:

$$\partial_s \phi(s, t) = s(s^2 + t^2)^{-\frac{1}{2}}, \quad \partial_t \phi(s, t) = t(s^2 + t^2)^{-\frac{1}{2}}.$$

The area of the graph over Ω is given by the integral

$$\int_0^1 \int_0^1 \sqrt{1 + \frac{s^2}{s^2 + t^2} + \frac{t^2}{s^2 + t^2}} \, ds dt = \sqrt{2}.$$

The integral of f over the graph S reads as follows:

$$\begin{aligned} \iint_S f \, d\sigma &= \int_0^1 \int_0^1 s \cdot t \cdot \sqrt{s^2 + t^2} \cdot \sqrt{1 + \frac{s^2}{s^2 + t^2} + \frac{t^2}{s^2 + t^2}} \, ds dt \\ &= \sqrt{2} \int_0^1 \int_0^1 s \cdot t \cdot \sqrt{s^2 + t^2} \, ds dt. \end{aligned}$$

We simplify it further:

$$\int_0^1 \int_0^1 s \cdot t \cdot (s^2 + t^2)^{\frac{1}{2}} \, ds dt$$

$$\begin{aligned}
&= \int_0^1 t \cdot \frac{1}{3} \int_0^1 \partial_s (s^2 + t^2)^{\frac{3}{2}} ds dt \\
&= \int_0^1 t \cdot \frac{1}{3} \left[(s^2 + t^2)^{\frac{3}{2}} \right]_{s=0}^{s=1} dt \\
&= \int_0^1 t \cdot \frac{1}{3} \left((1 + t^2)^{\frac{3}{2}} - t^3 \right) dt = \frac{1}{3} \int_0^1 t \cdot (1 + t^2)^{\frac{3}{2}} - t^4 dt
\end{aligned}$$

Next,

$$\begin{aligned}
&\int_0^1 t \cdot (1 + t^2)^{\frac{3}{2}} - t^4 dt \\
&= \int_0^1 \frac{1}{5} \partial_t \left((1 + t^2)^{\frac{5}{2}} \right) - \partial_t \left(\frac{1}{5} t^5 \right) dt \\
&= \frac{1}{5} \left[(1 + t^2)^{\frac{5}{2}} \right]_{t=0}^{t=1} - \frac{1}{5} [t^5]_{t=0}^{t=1} = \frac{1}{5} \left(\sqrt{2^5} - 1 - 1 \right).
\end{aligned}$$

In summary,

$$\iint_S f d\sigma = \sqrt{2} \cdot \frac{1}{3} \cdot \frac{1}{5} \left(\sqrt{2^5} - 1 - 1 \right) = \frac{\sqrt{2}}{15} \cdot \left(\sqrt{2^5} - 2 \right) = \frac{2}{15} \cdot \left(4 - \sqrt{2} \right).$$

Exercise 6 The parameterization

$$\Phi : [0, 2\pi) \times (0, 1) \rightarrow \mathbb{R}^3, \quad (\theta, z) \mapsto ((1 + z) \cos(\theta), (1 + z) \sin(\theta), z)$$

describes a surface S . Find the surface area of S .

Solution 6 We compute the derivatives of Φ :

$$\partial_\theta \Phi(\theta, z) := \begin{pmatrix} -(1 + z) \sin(\theta) \\ (1 + z) \cos(\theta) \\ 0 \end{pmatrix} \quad \partial_z \Phi(\theta, z) := \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 1 \end{pmatrix}.$$

The cross product $\partial_\theta \Phi(\theta, z) \times \partial_z \Phi(\theta, z)$ equals

$$\begin{pmatrix} -(1 + z) \sin(\theta) \\ (1 + z) \cos(\theta) \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 1 \end{pmatrix} = \begin{pmatrix} (1 + z) \cos(\theta) \\ (1 + z) \sin(\theta) \\ -(1 + z) \end{pmatrix}$$

The norm of cross product is

$$\begin{aligned}
\|\partial_\theta \Phi(\theta, z) \times \partial_z \Phi(\theta, z)\| &= \|((1 + z) \cos(\theta), (1 + z) \sin(\theta), -(1 + z))\| \\
&= \sqrt{(1 + z)^2 \cos^2(\theta) + (1 + z)^2 \sin^2(\theta) + (1 + z)^2} \\
&= \sqrt{2(1 + z)^2}
\end{aligned}$$

$$= \sqrt{2} \cdot (1 + z).$$

We can now compute the surface area:

$$\begin{aligned} \iint_S 1 d\sigma &= \int_0^1 \int_0^{2\pi} 1 \cdot \sqrt{2}(1 + z) d\theta dz \\ &= 2\pi \cdot \sqrt{2} \cdot \left[z + \frac{1}{2}z^2 \right]_{z=0}^{z=1} \\ &= 2\pi \cdot \sqrt{2} \cdot \frac{3}{2} \\ &= 3\sqrt{2} \cdot \pi. \end{aligned}$$

Exercise 7 Let $f(x_1, x_2, x_3) = x_1x_2 + x_3^2$ and consider the surface

$$S := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 < x_3 < 1, x_1^2 + x_2^2 = x_3^2 \}$$

Find a parameterization of S and compute the surface integral $\iint_S f d\sigma$.

Solution 7 We define the following parameterisation for the surface S :

$$\Phi : [0, 2\pi) \times (0, 1) \mapsto S : (\theta, z) \mapsto (z \cos \theta, z \sin \theta, z)$$

with

$$\|\partial_\theta \Phi \times \partial_z \Phi\| = \left\| \begin{pmatrix} -z \sin \theta \\ z \cos \theta \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} z \cos \theta \\ z \sin \theta \\ -z \end{pmatrix} \right\| = \sqrt{2}z$$

Now the surface integral is given by:

$$\begin{aligned} \iint_S f dS &= \int_0^{2\pi} \int_0^1 (z^2 \cos \theta \sin \theta + z^2) \sqrt{2}z dz d\theta \\ &= \int_0^{2\pi} \int_0^1 \sqrt{2}z^3 dz d\theta + \int_0^{2\pi} \cos \theta \sin \theta d\theta \int_0^1 \sqrt{2}z^3 dz \\ &= 2\pi \left[\frac{\sqrt{2}}{4} z^4 \right]_0^1 + \int_0^{2\pi} \frac{1}{2} \sin 2\theta d\theta 2\pi \left[\frac{\sqrt{2}}{4} z^4 \right]_0^1 \\ &= \pi \frac{2\sqrt{2}}{4} + \left[-\frac{1}{4} \cos 2\theta \right]_0^{2\pi} \pi \frac{2\sqrt{2}}{4} \\ &= \pi \frac{2\sqrt{2}}{4} + \left(-\frac{1}{4} - -\frac{1}{4} \right) \pi \frac{2\sqrt{2}}{4} = \pi \frac{\sqrt{2}}{2}. \end{aligned}$$