October 30

Stokes' theorem and orientations

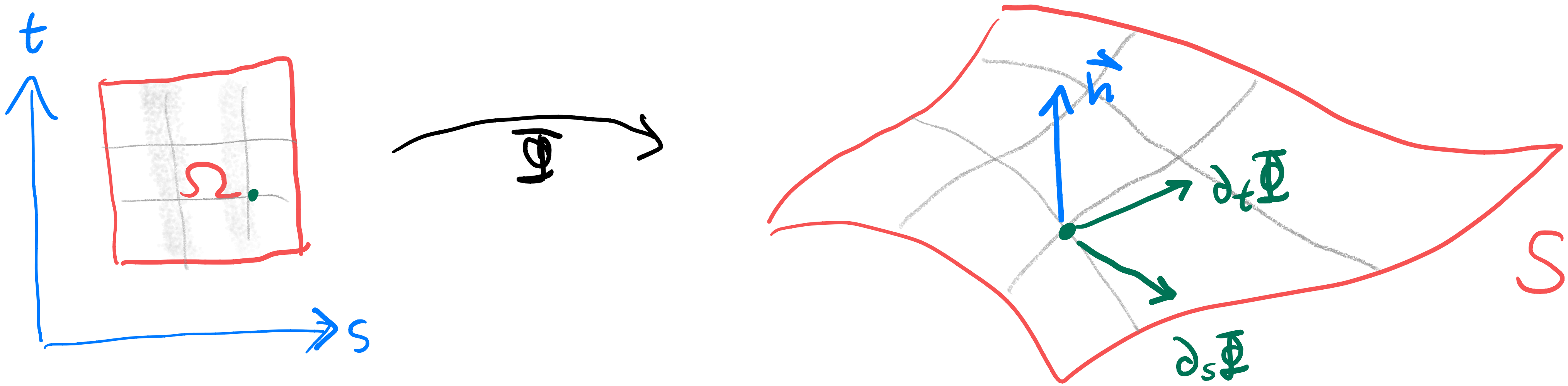
The divergence theorem generalizes from 2D domains to 3D volumes in a straight-forward manner

Green's theorem generalizes from 2D domains to 3D surfaces, where we call it Stokes' theorem

Important concept: orientation of surfaces

Theory

Let $\Omega \subseteq \mathbb{R}^2$ be a domain and $\Phi : \Omega \to \mathbb{R}^3$ be a regular parametrization of some surface S



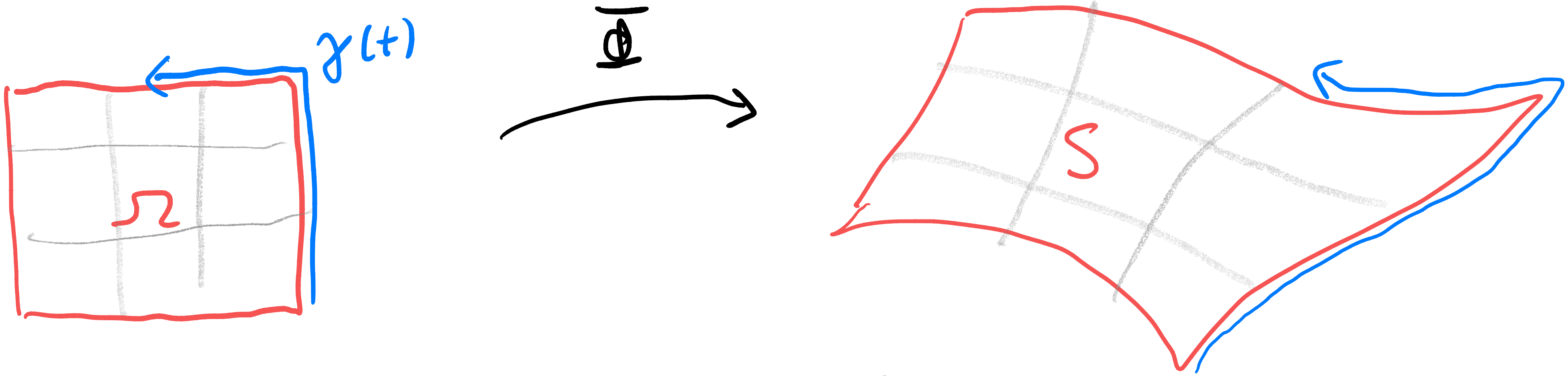
The parametrization determines a unit normal direction

$$\vec{n} = \partial_s \Phi \times \partial_t \Phi \,/\, \| \partial_s \Phi \times \partial_t \Phi \|$$

The surface boundary ∂S is a curve C

Assume that Φ maps $\partial\Omega$ onto ∂S bijectively

Let $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ be a (piecewise) regular parametrization of the boundary $\partial\Omega$ in ccw orientation



We obtain the parametrization of ∂S:

$$\Phi \circ \gamma : [a,b] \longrightarrow \mathbb{R}^3$$

Stokes theorem

Let the geometric setting be as above

Let $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is a differentiable vector field

$$\iint_S \operatorname{curl} F \, d\sigma = \oint_C \vec{F} \, dl$$

where S and C have orientation given by the parametrization Φ.

$$\iint_S \operatorname{curl} F \cdot \vec{n} \, d\sigma = \oint_C \vec{F} \cdot \vec{T} \, dl$$

surface unit normal induced by parametrization

unit tangent vector in direction induced by parametrization

- The <u>sign</u> of the surface integral depends on which unit normal we use, which in turn depends on the parametrization
- The <u>sign</u> of the boundary integral depends on the choice of direction for C, which in turn depends on the parametrization
- For a fixed parametrization Φ, the signs match so that Stokes' theorem holds

For practical computations:

- We can use any other parametrization of the surface, as long as it gives the same unit normal
- " " " " " " " " " " " " boundary curve, " " " " " " " " " " " " " " tangent direction

Recap

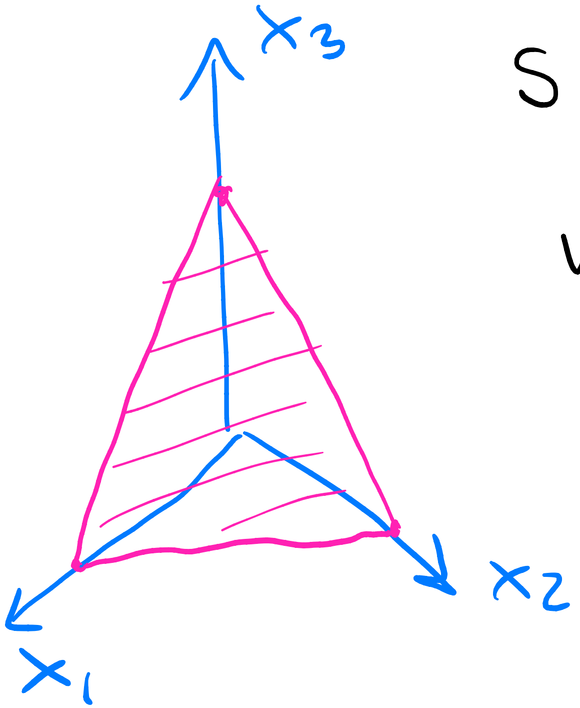
$$\iint_S \operatorname{curl} F \cdot \vec{n}\, d\sigma := \iint_\Omega \operatorname{curl} \vec{F}(\Phi(s,t)) \cdot \vec{n}(\Phi(s,t)) \, \|\partial_s \Phi \times \partial_t \Phi\| \, ds dt$$

$$= \iint_\Omega \operatorname{curl} \vec{F}(\Phi(s,t)) \cdot (\partial_s \Phi \times \partial_t \Phi) \, ds dt$$

$$\oint_C \vec{F}\, d\ell = \oint_C \vec{F} \cdot \vec{T}\, d\ell = \int_a^b \vec{F}(\Phi \circ \gamma) \cdot (\Phi \circ \gamma)' \, dt$$

Example 1

The triangle surface S is



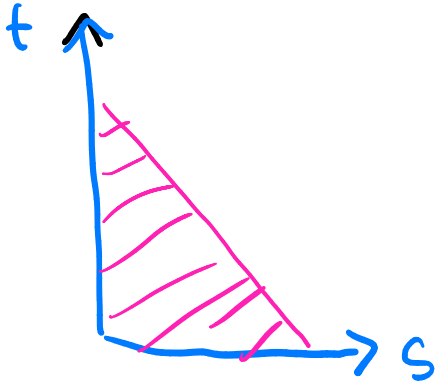
$$S = \{ \vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1, \quad x_1, x_2, x_3 > 0 \}$$

What is a parametrization of S?

$$\Omega = \{ (s,t) \in \mathbb{R}^2 \mid s,t > 0, \quad s+t < 1 \}$$

$$\Phi : \Omega \to \mathbb{R}^3$$

$$(s,t) \mapsto (s, t, 1-s-t)$$



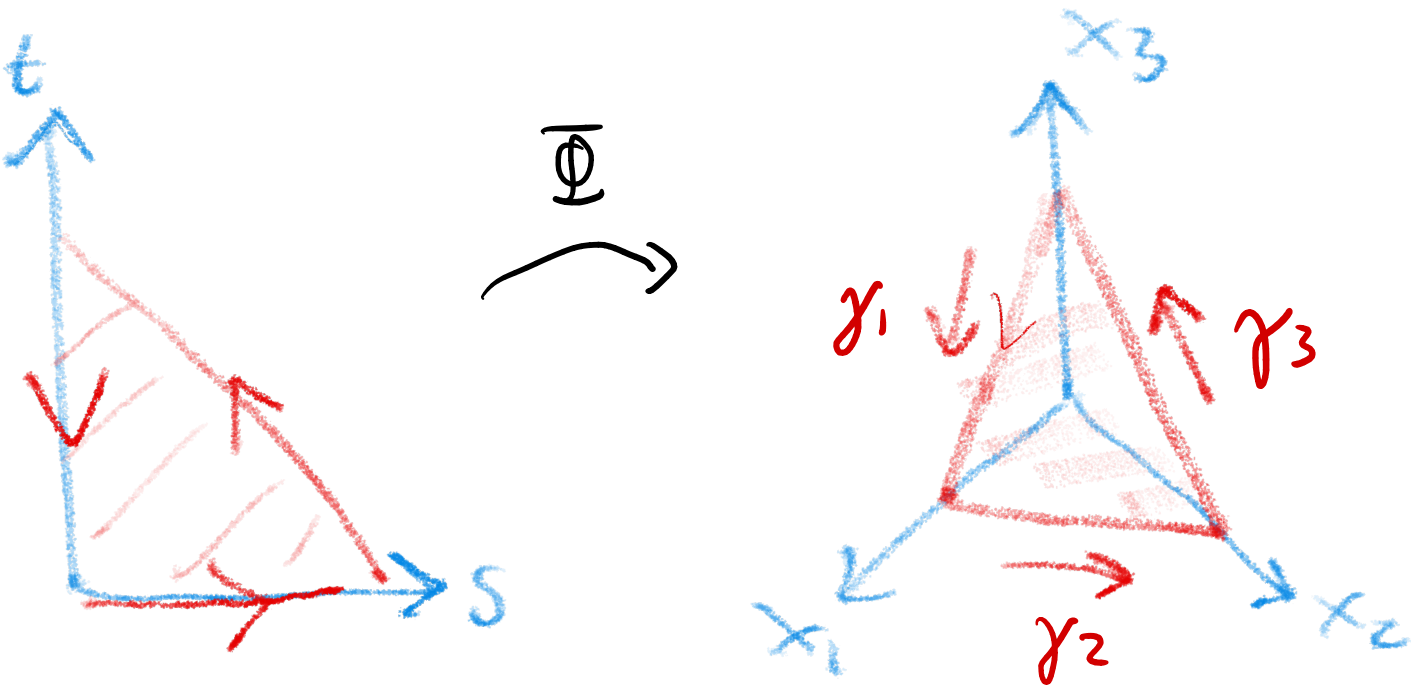
Visually, we map s and t to x_1 and x_2, respectively, and map the origin to $(0,0,1)$

$$\partial_s \Phi = (1, 0, -1) \qquad \partial_s \Phi \times \partial_t \Phi = (1, 1, 1)$$

$$\partial_t \Phi = (0, 1, -1)$$

Hence, $\vec{n} = \dfrac{\partial_s \Phi \times \partial_t \Phi}{\| \partial_s \Phi \times \partial_t \Phi \|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$

We parametorize the boundry curve of ∂S in 3 pieces



The ccw parametrization of $\partial \Omega$ gives a parametrization of ∂S

$\gamma_1 : [0,1] \to \mathbb{R}^3, \quad r \mapsto (r, 0, 1-r)$

$\gamma_2 : [0,1] \to \mathbb{R}^3, \quad r \mapsto (1-r, r, 0)$

$\gamma_3 : [0,1] \to \mathbb{R}^3, \quad r \mapsto (0, 1-r, r)$

This is enough to compute the curve integrals along $C = \partial S$

Let's consider the vector field

$$\vec{F}(x_1, x_2, x_3) = \left(x_1 + x_2^2, \ x_2 + x_3^2, \ x_3 + x_1^2 \right)$$

We need the curl

$$\operatorname{curl} F(x_1, x_2 x_3) = \left(-2x_3, \ -2x_1, \ -2x_2 \right)$$

According to Stokes' theorem

$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, d\sigma = \oint_C \vec{F} \cdot \vec{\jmath} \, d\ell$$

given by parametrization

We find the curve integral using the surface integral:

$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, d\sigma = \iint_\Omega \operatorname{curl} \vec{F}(\Phi(s,t)) \cdot \left(\partial_s \Phi(s,t) \times \partial_t \Phi(s,t) \right) ds\,dt$$

$$= \int_0^1 \int_0^{1-s} -2 \cdot (1-s-t, s, t) \cdot (1,1,1) \, dt\, ds$$

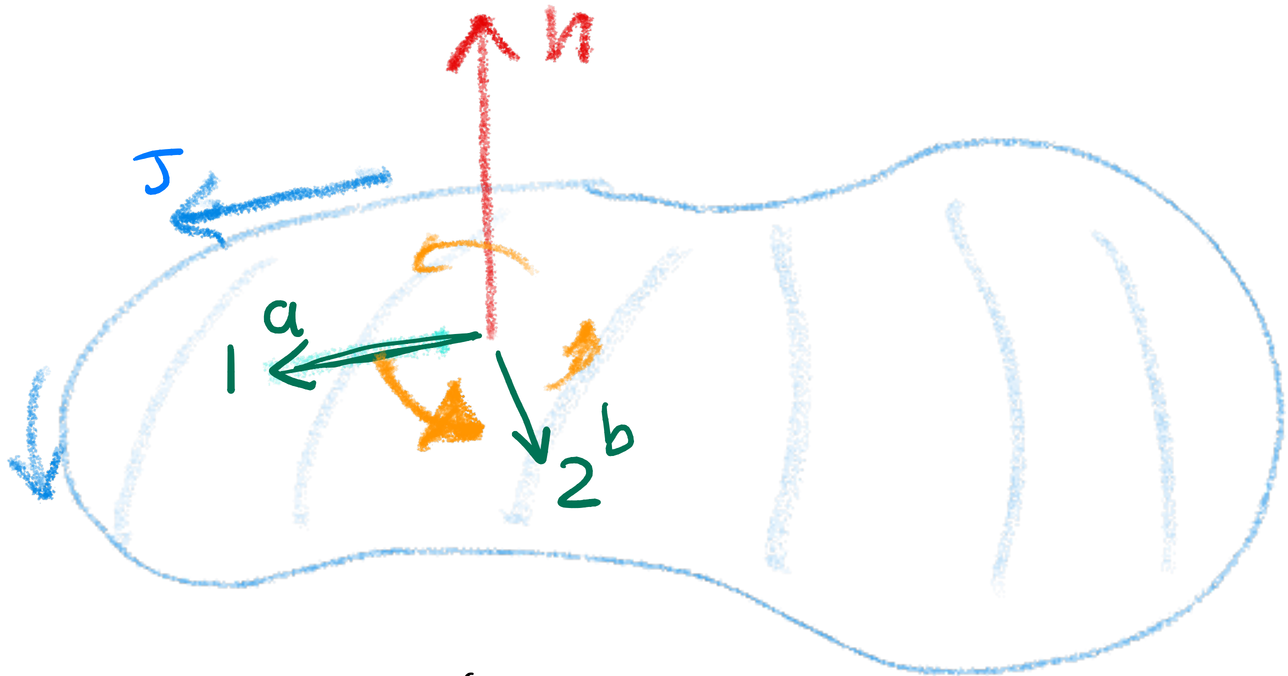
$$= -2 \int_0^1 \int_0^{1-s} 1-s-t+s+t \, ds\,dt = -2 \int_0^1 \int_0^{1-s} 1 \, ds\,dt$$

$$= -1$$

Exercise: compute $\oint_C F \cdot \vec{\jmath}$ directly

More theory

Can choose the orientation of the surface (that is, the choice of unit normal) and the orientation of the boundary (that is, the choice of direction/unit tangent) such that Stokes' theorem holds without using a parametrization?



n is in the direction of $a \times b$

If n and J match like this, then Stokes' theorem holds.

If not, then signs will mismatch

Suppose you walk on the surface with your head in the direction of n. We walk along the boundary such our left arm is free. Then this gives the direction of the boundary.

Along the boundary we check for which vector $\vec{b}$ we have $\vec{J} \times \vec{b} = \vec{n}$. If it points into the surface, then the orientation of S and ∂S match.

$\boxed{\text{So we have several geometric criteria to figure out whether the orientations of } S \text{ and } \partial S \text{ match and Stokes' theorem applies}}$

Moving on to the next example, suppose we have radial coordinates