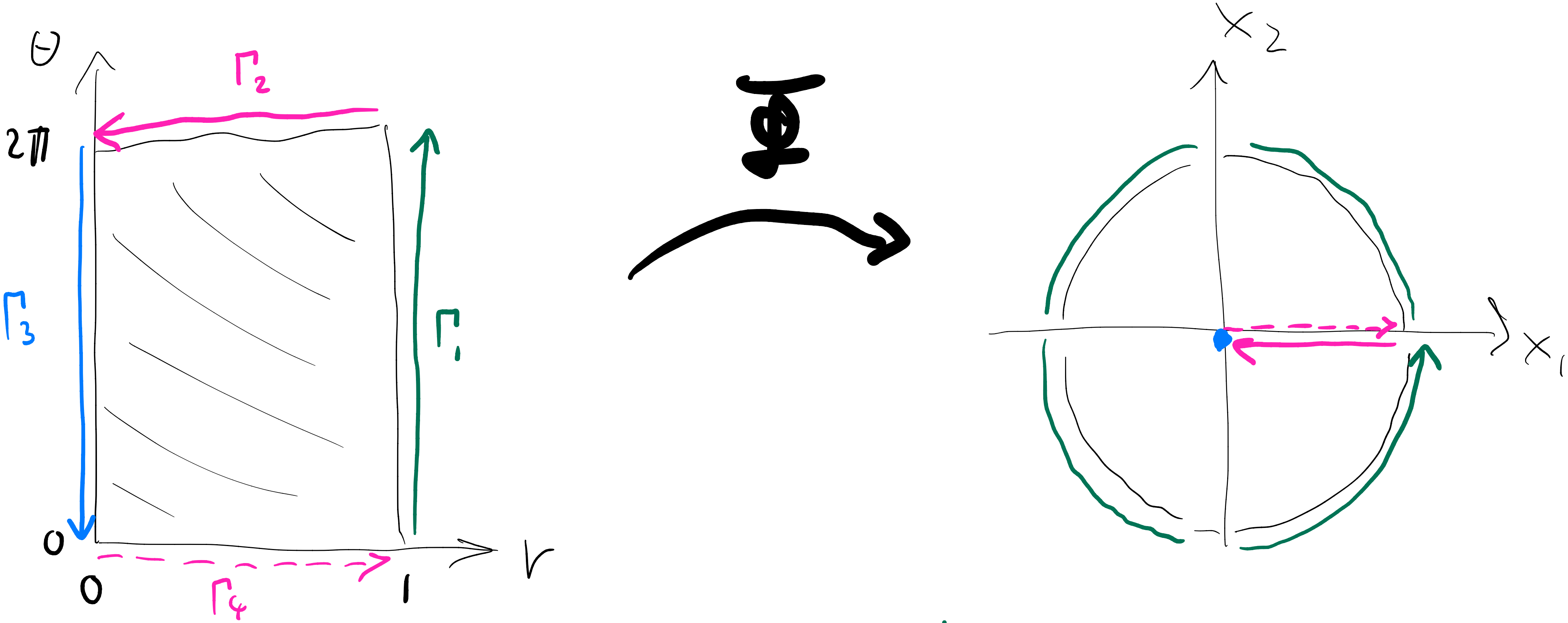
November 7

Stokes theorem

Fourier Analysis

Yet another look at radial coordinates



We observe:

- right side is what we want: parametorization of disk boundary
- The integrals along the lines corresponding upper and lower part of the rectangle cancel out
  on the physical disk, it's the same line but in different directions
- The integral along the left-side is zero

Hence only the part corresponding to the physical boundary remains

If $\gamma : [0,1] \to \mathbb{R}^2$ parametorizes the boundary of the parameter domain

$$\Omega = (0,1) \times (0, 2\pi)$$

Then $\Phi \circ \gamma$ maps the 4 boundary pieces into the disk

$$\int_{\Phi \circ \gamma} \vec{F}\, dl = \int_{\Phi \circ \Gamma_1} \vec{F}\, dl + \int_{\Phi \circ \Gamma_2} \vec{F}\, dl + \int_{\Phi \circ \Gamma_3} \vec{F}\, dl + \int_{\Phi \circ \Gamma_4} \vec{F}\, dl$$

physical boundary integral

$= 0$

cancel out, $= 0$

Summary of 3D vector analysis

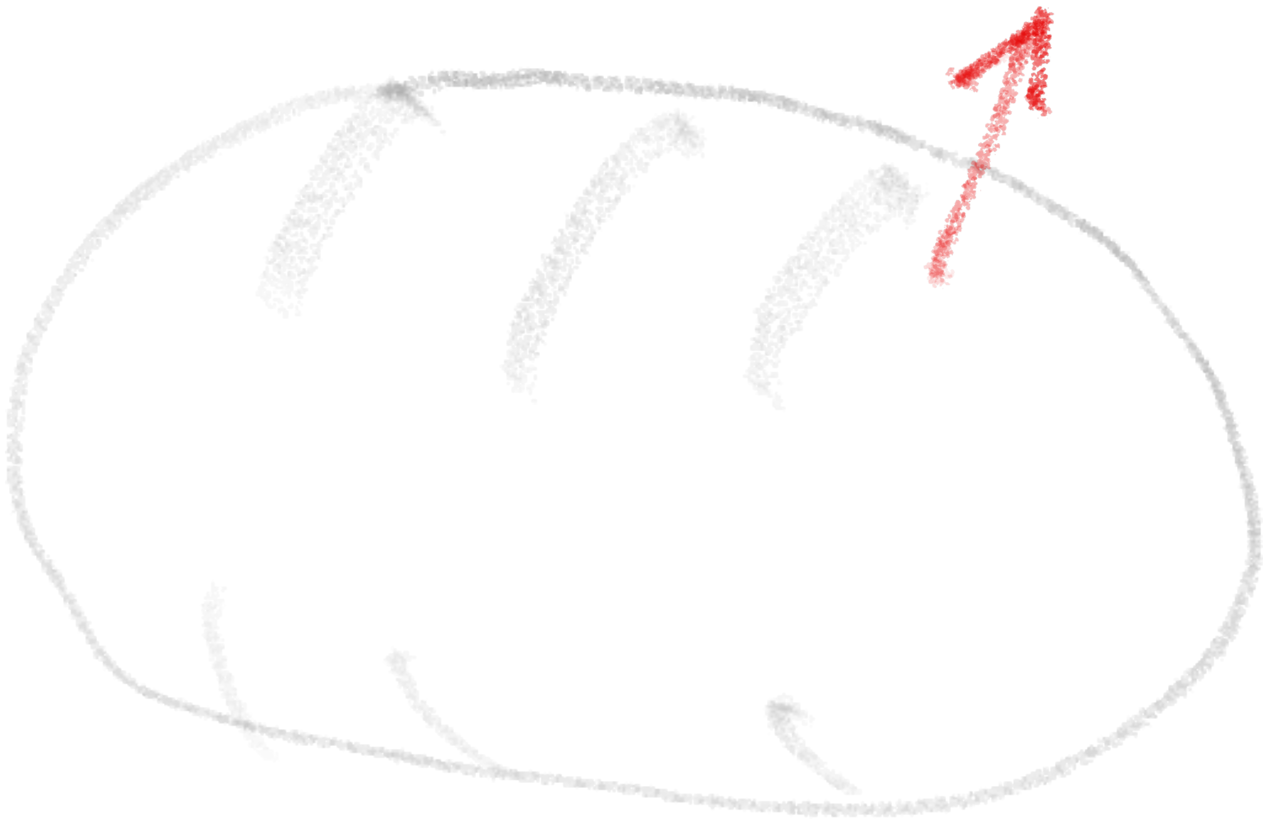
- Divergence theorem in 3D
- Stokes' theorem in 3D

While the divergence theorem and Green's theorem in 2D are similar, the divergence theorem and Stokes' theorem in 3D are quite different.

Divergence in 3D

only for closed surfaces without boundary

For example: soap bubble



The surface splits 3D space into inside and outside

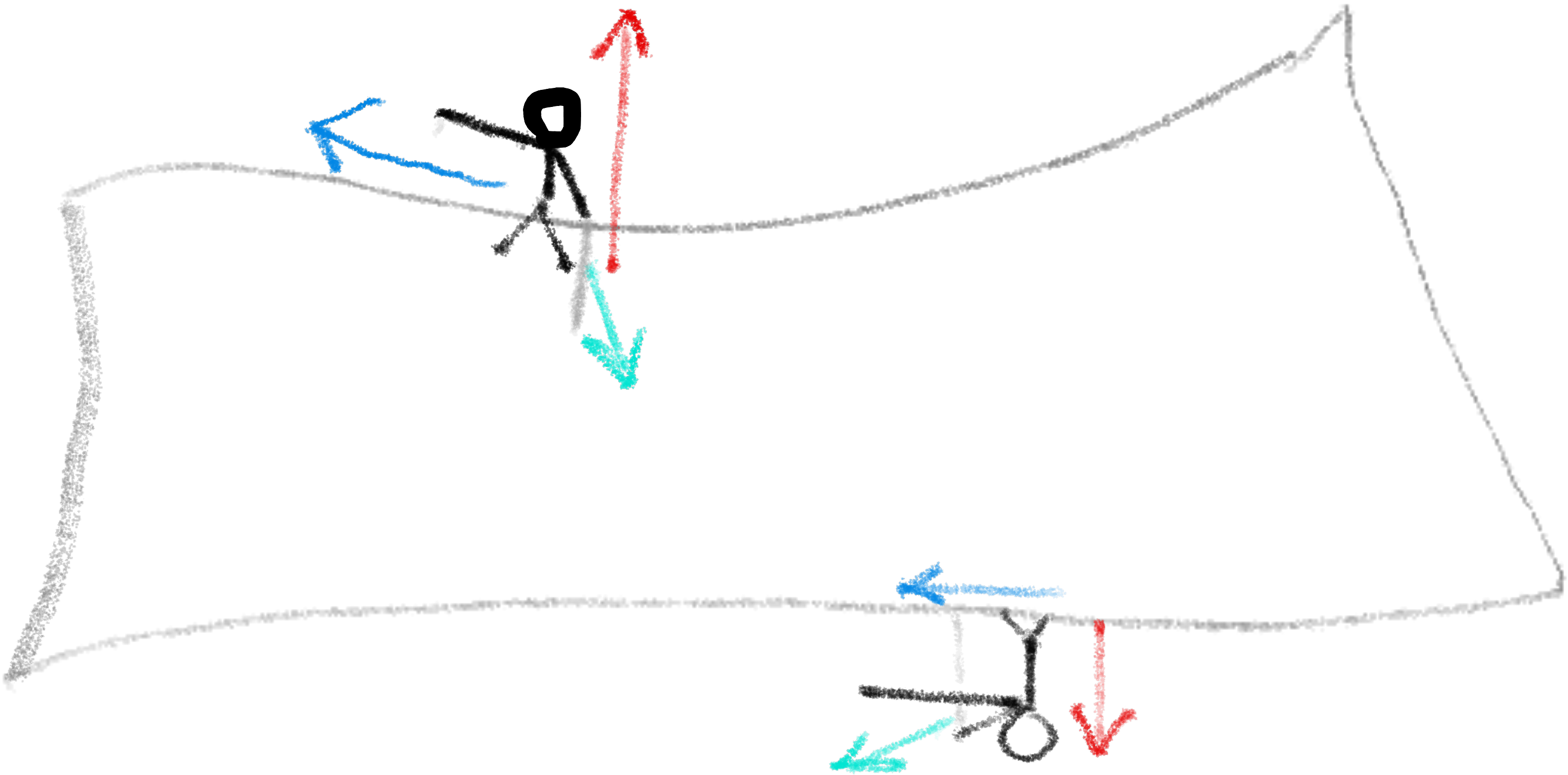
Hence, we can determine* which unit normal points outside

* in principle

Stokes' theorem in 3D

For surfaces in 3D with boundary. No inside and outside.

Whatever orientation (surface normal) we choose for the surface, the orientation of the boundary (tangent direction) must match.



Coda: change of variables, revisited

Suppose that $X, Y \subseteq \mathbb{R}^2$ are two domains and that $\Phi : X \to Y$ is differentiable and injective

If $f : Y \to \mathbb{R}$ is a scalar function, then

$$\iint_{\Phi(X)} f(y)\, dy_1 dy_2 = \iint_X f(\Phi(x)) \cdot |\det D\Phi(x)|\, dx_1 dx_2$$

(Integral transformulation formula)

Here, $$|\det D\Phi(x)| = \left|\det\begin{pmatrix} \partial_1\Phi_1 & \partial_2\Phi_1 \\ \partial_1\Phi_2 & \partial_2\Phi_2 \end{pmatrix}\right|$$

This formula is similar to the formula for surface integrals

$$\iint_S f\, d\sigma = \iint_\Omega f(\Phi(s,t))\, \| \partial_s\Phi \times \partial_t\Phi \|\, ds\,dt$$

conceptually, the factor measures how Φ "stretches" the area

Suppose that $\Phi : \Omega \to \mathbb{R}^3$ maps into x_1x_2-plane:

$$\Phi(s,t) = (\Phi_1(s,t),\ \Phi_2(s,t),\ 0)$$

Let $S := \Phi(\Omega)$ be the surface in the x_1x_2-plane. Then

$$\iint_S f(x_1, x_2, \underset{\substack{\uparrow \\ x_3 = 0 \\ \text{over } S}}{x_3})\, d\sigma = \iint_\Omega f(\Phi_1(s,t), \Phi_2(s,t), 0) \underbrace{\| \partial_s \Phi \times \partial_t \Phi \|}_{\circledast =} ds\,dt$$

$$\circledast = \left\| \begin{pmatrix} 0 \\ 0 \\ \partial_s \Phi_1 \partial_t \Phi_2 - \partial_s \Phi_2 \partial_t \Phi_1 \end{pmatrix} \right\| = \left| \partial_s \Phi_1 \partial_t \Phi_2 - \partial_s \Phi_2 \partial_t \Phi_1 \right|$$

$$= \left| \det \begin{pmatrix} \partial_s \Phi_1 & \partial_t \Phi_1 \\ \partial_s \Phi_2 & \partial_t \Phi_2 \end{pmatrix} \right|$$

using definition of cross product and that the third coordinate of Φ is constant zero

Moral of the story:

The formula for surface integrals generalizes the change of variables formula (changement de variable)

Second part of Analysis III

Fourier Analysis & Applications

- Extension of Analysis 1
- Preparation for signal processing
  - representation of signals by their frequencies
  - Application: audio, image processing
- Solution of ODEs & PDEs

ordinary differential equations

partial differential equations

1. Distribution theory

We develop a theory of <u>generalized functions</u>

1.1. Instead of point evaluations, we can "probe" functions via integration against other functions.

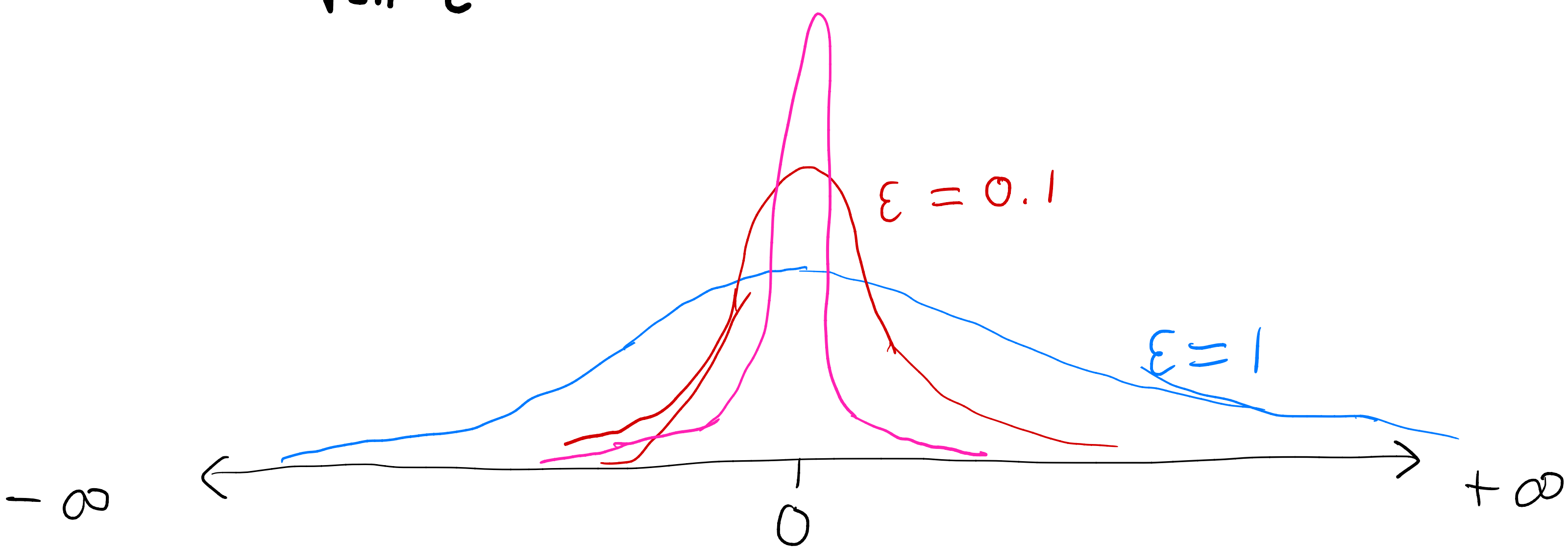
If $\varphi : \mathbb{R} \to \mathbb{R}$ is a function, for example, we can "probe" it with an integral against some function $g : \mathbb{R} \to \mathbb{R}$

$$\int_{\mathbb{R}} \varphi(x)\, g(x)\, dx$$

Interpretation:
How does φ react with the signal g?

For example, consider the Gaussian bump

$$g_\varepsilon(x) := \frac{1}{\sqrt{2\pi}\,\varepsilon} e^{-x^2/2\varepsilon^2} \qquad \varepsilon \text{ is a parameter}$$



First, we show that $g_\varepsilon(x)$ has integral one:

$$\int_{\mathbb{R}} g(x)\,dx = \frac{1}{\sqrt{2\pi}\,\varepsilon} \int_{\mathbb{R}} e^{-x^2/2\varepsilon^2}$$

$$\left[\begin{array}{l} \text{substitute } u = \dfrac{x}{\sqrt{2}\,\varepsilon} \\ \qquad du = \dfrac{dx}{\sqrt{2}\,\varepsilon} \end{array} \right]$$

$$= \frac{1}{\sqrt{2\pi}\,\varepsilon} \int_{\mathbb{R}} e^{-u^2} \sqrt{2}\,\varepsilon \, du = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-u^2} du$$

Exercise $\rightarrow$ $= 1$

Interpretation Since $g_\varepsilon(x)$ has integral 1, $\int_{\mathbb{R}} \varphi(x)\, g_\varepsilon(x)\, dx$ is an average of function values of φ with some "weight" g_ε

As $\varepsilon \to 0$, the weight will be more concentrated around $x = 0$.

We think of it as an averaged point evaluation.

If φ is continuous, then $\lim\limits_{\varepsilon \to 0} \int_{\mathbb{R}} \varphi(x)\, g_\varepsilon(x)\, dx = \varphi(0)$

In many practical applications, we don't have access to point values of a signal, but only to "averages" like this or similar

Q: Does there exist a function $g_0 : \mathbb{R} \to \mathbb{R}$ with integral 1 such that $\int_{\mathbb{R}} g_0(x)\, \varphi(x)\, dx = \varphi(0)$?

A: No, but there exists a generalized function with that property

Specifically, the Dirac-Delta at zero is the functional

$$\delta_0 : C^0(\mathbb{R}) \to \mathbb{R}, \qquad f \to f(0)$$

This is a functional, i.e., a real-valued function of functions, mapping each function to its value at $x = 0$.

Moreover, its "integral" is the value of the constant function $f(x) \equiv 1$, which is $\delta_0(f) = 1$.

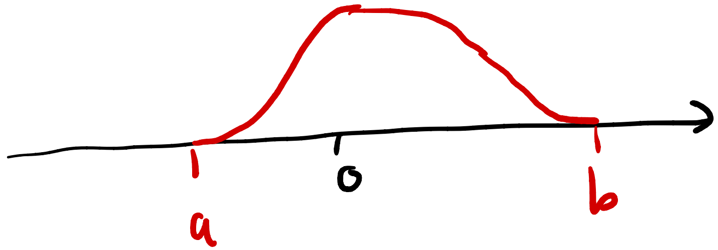
$$\left[\int_{\mathbb{R}} \delta_0 \, dx = \int_{\mathbb{R}} \delta_0 \cdot 1 \, dx = \lim_{\varepsilon \to 0} \int g_\varepsilon(x) 1 \, dx = 1 \right]$$

purely heuristic

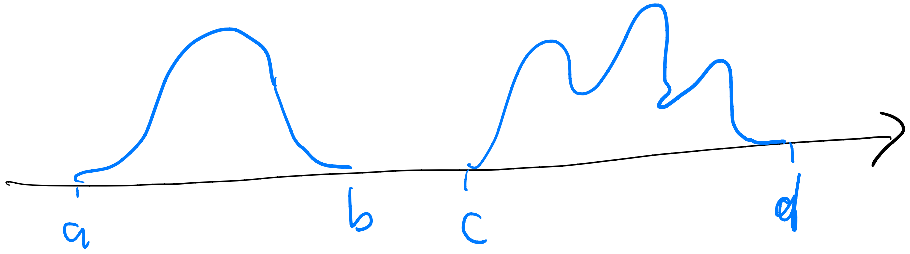
Putting this on a more rigorous footing, we introduce distributions.

2. Definitions

In what follows, the support $\operatorname{supp} f$ of a function $f: \mathbb{R} \to \mathbb{R}$ is the closure of $\{ x \in \mathbb{R} \mid f(x) \neq 0 \}$



$\text{supp } f = [a,b]$

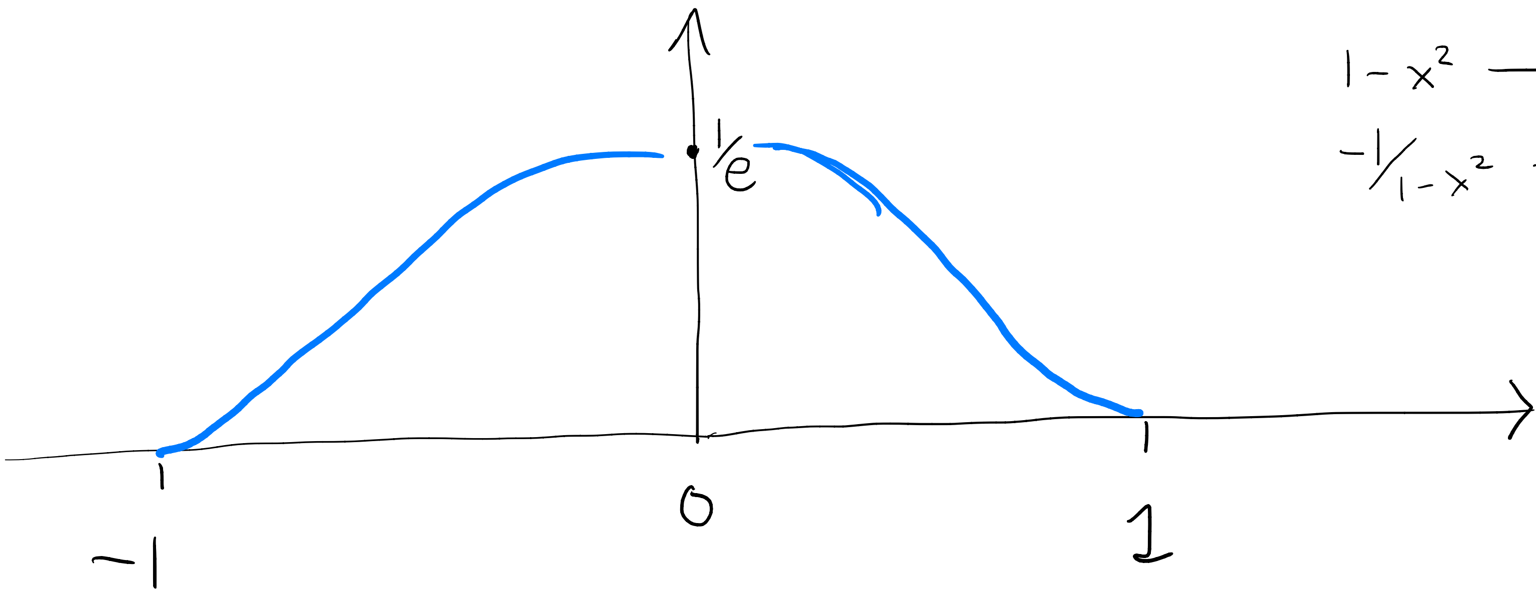


$\text{supp } f = [a,b] \cup [c,d]$

<u>Example</u>: Consider the function

$$\varphi(x) = \begin{cases} e^{-1/1-x^2} & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

The support of φ is the interval $[-1, 1]$, obviously. What is not as obvious: the function φ has derivatives of all orders, that is, $\varphi \in C^\infty(\mathbb{R})$



As $x \to \pm 1$

$1 - x^2 \to 0$

$-1/_{1-x^2} \to -\infty$

We check φ is differentiable: $\varphi' = 0$ outside $[-1, 1]$

$$\varphi'(x) = e^{-1/_{1-x^2}} \cdot (-1)(1-x^2)^{-2} \cdot (-2x)$$

$$= e^{-1/_{1-x^2}} \frac{2x}{(1-x^2)^2} \qquad \text{over } (-1, 1)$$

As $x \to \pm 1$, $e^{-1/_{1-x^2}}$ goes to zero, faster, than $\frac{2x}{(1-x^2)^2}$ blows up (goes to ∞)

Hence

$$\varphi'(x) = \begin{cases} e^{-1/(1-x^2)} \cdot \dfrac{2x}{(1-x^2)^2} & \text{if } x \in (-1,1) \\ 0 & \text{if } x \notin (-1,1) \end{cases}$$

Similar arguments apply for the higher derivatives ...

<u>Notice:</u> φ has compact support, that is, $\operatorname{supp} \varphi$ is within a bounded interval, but the function is smooth (differentiable infinitely often)

We let

$$\mathcal{D} := C_c^\infty(\mathbb{R}) = \{ \varphi \in C^\infty(\mathbb{R}) \mid \operatorname{supp} \varphi \subseteq \text{bounded interval} \}$$

We let $\mathcal{D}'$ be the set of distributions over $\mathbb{R}$, which is the set of linear continuous functionals over $\mathcal{D}$

Explicitly, $f \in \mathcal{D}'$ means $f : \mathcal{D} \to \mathbb{R}$ such that

- f is finite
- f is linear :

$$\forall \alpha, \beta \in \mathbb{R}, \quad \varphi, \psi \in \mathcal{D} : \qquad f(\alpha\varphi + \beta\psi) = \alpha f(\varphi) + \beta f(\psi)$$

- f is "continuous", that is, "f changes little if the input changes little"

Explicitly, for each $[a,b] \subseteq \mathbb{R}$ there exist $C > 0$ and $k \in \mathbb{N}_0$ such that

$$\forall \varphi \in \mathcal{D} : \ \operatorname{supp}(\varphi) \subseteq [a,b] \qquad |f(\varphi)| \leq C \sum_{0 \leq i \leq k} \max_{x \in \mathbb{R}} |\partial^i \varphi(x)|$$

We will also write $\langle f, \varphi \rangle := f(\varphi)$

3. Examples

1) Let f be any integrable function over $\mathbb{R}$. Then

$$\langle f, \varphi \rangle := \int_{\mathbb{R}} f(x)\, \varphi(x)\, dx$$

is a distribution. Indeed:

- $\langle f, \varphi \rangle$ is always finite
- $\langle f, \varphi \rangle$ is linear in φ

$$\langle f, \alpha\varphi + \beta\psi \rangle = \alpha \langle f, \varphi \rangle + \beta \langle f, \psi \rangle$$

- $\langle f, \varphi \rangle$ is continuous:

$$|\langle f, \varphi \rangle| = \left| \int_{\mathbb{R}} f(x)\, \varphi(x)\, dx \right| \leq \underbrace{\left| \int_{\mathbb{R}} |f(x)|\, dx \right|}_{=: C} \cdot \max_{x \in \mathbb{R}} |\varphi(x)|$$

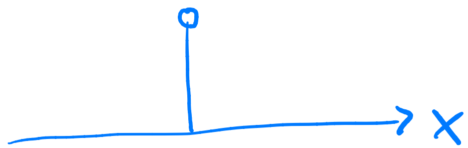
Here, (f, φ) is the preferable notation over $f(\varphi)$, even though in practice there is no ambiguity what is meant

Distributions that are integrable functions are called <u>regular</u>

[In fact, locally integrable functions work too]

but not all distributions are regular. In that sense, distributions are also called "generalized functions"

2) Dirac-Delta / Dirac pulse / point mass



$$\delta_0 : \mathcal{D} \to \mathbb{R}, \qquad \varphi \longmapsto \varphi(0)$$

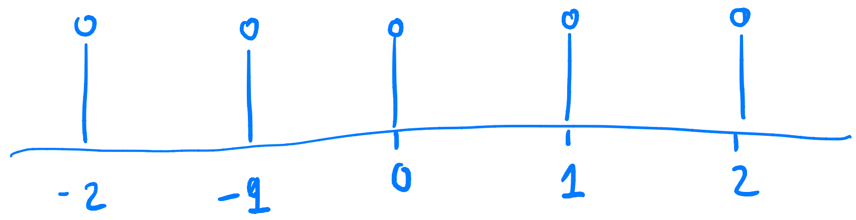
Obviously, δ_0 is finite and linear. Moreover

$$\forall \varphi \in \mathcal{D} : \qquad |\delta_0(\varphi)| = |\varphi(0)| \leq \max_{x \in \mathbb{R}} |\varphi(x)|$$

Here, $C = 1$

3) Dirac comb

$$\Delta_1(\varphi) := \sum_{n\in\mathbb{Z}} \varphi(n)$$



- For each $\varphi \in D$, the value $\Delta_1(\varphi)$ is finite, because the support of φ is a bounded interval, say, $[a,b]$, which can only contain finitely many integers. Hence

$$\Delta_1(\varphi) := \sum_{n\in\mathbb{Z}} \varphi(n) = \sum_{\substack{n\in\mathbb{Z} \\ n\in[a,b]}} \varphi(n) < \infty$$

is a sum of only finitely many terms.

The interval $[a,b]$ will always depend on φ

- $\Delta_1(\varphi)$ is linear because for all $\alpha, \beta \in \mathbb{R}$ and $\varphi, \psi \in \mathcal{D}$, we have $\alpha\varphi + \beta\psi \in \mathcal{D}$ [meaning, $\mathcal{D}$ is a vector space], so $\alpha\varphi + \beta\psi$ has support within some interval $[a,b]$

which also includes the support of φ and Ψ.