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Fourier transform

Coda of Fourier series

As we have seen, given $f: \mathbb{R} \to \mathbb{R}$ with period T the Fourier coefficients are

$$a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{2\pi n}{T} x \right) dx , \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi n}{T} x \right) dx , \qquad n = 1, 2, 3, \ldots$$

Fourier series

$$F f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T} x \right) + b_n \sin\left(\frac{2\pi n}{T} x \right)$$

Convergence? Dirichlet theorem!

Complex Fourier coefficients

$$c_n = \frac{1}{T}\int_0^T f(x)\, e^{-i\frac{2\pi n}{T}x}\,dx, \qquad n = \dots,-2,-1,0,1,2,\dots$$

Complex representation of Fourier series

$$Ff(x) = \sum_{n\in\mathbb{Z}} c_n\, e^{i\frac{2\pi n}{T}x}$$

How to switch from the real representation to complex representation?

$$c_0 = a_0/2$$

$$c_n e^{i\frac{2\pi n}{T}x} = c_n \cos\left(\frac{2\pi n}{T}x\right) + c_n \sin\left(\frac{2\pi n}{T}x\right)\cdot i$$

$$c_{-n} e^{-i\frac{2\pi n}{T}x} = c_{-n} \cos\left(\frac{2\pi n}{T}x\right) - c_{-n} \sin\left(\frac{2\pi n}{T}x\right)\cdot i$$

use cos even, sin odd

Thus

$$a_n = c_n + c_{-n} \qquad b_n = (c_{-n} - c_n)\,i$$

$$c_n = \frac{a_n}{2} - \frac{b_n}{2}i \qquad c_{-n} = \frac{a_n}{2} + \frac{b_n}{2}i$$

For the complex Fourier coefficients,

$$\int_0^T f(x)\, e^{-i\frac{2\pi n}{T}x}\, dx$$

needs to be computed. More generally, given $g : [a,b] \to \mathbb{C}$, how to compute

$$\int_a^b g(x)\, dx \quad ?$$

a) We decompose

$$\int_a^b g(x)\, dx = \int_a^b \operatorname{Re} g(x)\, dx + \int_a^b \operatorname{Im} g(x)\, dx \cdot i$$

that is, we integrate $\operatorname{Re} g$ and $\operatorname{Im} g$ separately.

b) Often more practically, we treat i like an unknown parameter when integrating / differentiating

Example:

$$\int_a^b e^{ix}\,dx = \int_a^b \left[\tfrac{1}{i} e^{ix}\right]' dx = \left[\tfrac{1}{i} e^{ix}\right]_{x=a}^{x=b}$$

$$= \tfrac{1}{i}\left(e^{bi} - e^{ai}\right)$$

Fourier transform

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a signal.

We define the Fourier transform:

$$\mathcal{F}(f)(\alpha) := \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\alpha t}\, dt$$

Physical interpretation: if $f : \mathbb{R} \to \mathbb{R}$ is a signal, amplitude $f(t)$ at time $t \in \mathbb{R}$, then $\hat{f} : \mathbb{R} \to \mathbb{C}$ describes the frequency component of frequency $\alpha \in \mathbb{R}$, $\hat{f}(\alpha)$ is the strength of that freqeency in the signal f.

Comparison with Fourier series

- In the Fourier series, we decompose a periodic signal into a discrete set of modes, each with component a_n or b_n

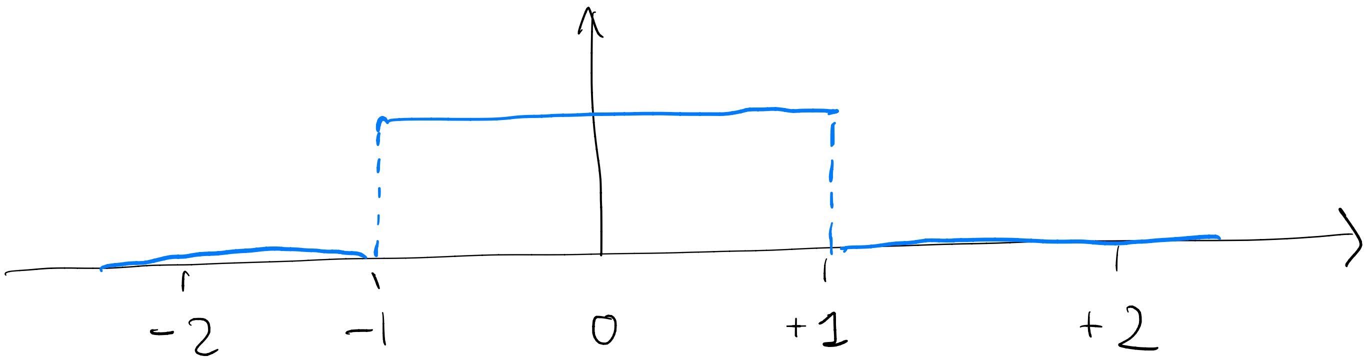
- In the Fourier transform, we decompose any signal into a continuous spectrum of modes, each with component $\hat{f}(\alpha)$.

Conventions: the literature has numerous diffent conventions how to exactly define the Fourier transform

Examples

1) Consider the signal $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$



We use the definition of the Fourier transform

$$\hat{f}(\alpha) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\alpha t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} e^{-i\alpha t}\, dt = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\alpha t}}{-\alpha i} \right]_{t=-1}^{t=+1}$$

$$= \frac{e^{i\alpha} - e^{-i\alpha}}{\sqrt{2\pi} \cdot \alpha i}$$

We can simplify this further:

$$= \frac{1}{\alpha\sqrt{2\pi}} \frac{e^{i\alpha} - e^{-i\alpha}}{i} = \frac{2}{\alpha\sqrt{2\pi}} \frac{e^{i\alpha} - e^{-i\alpha}}{2i} = \sqrt{\frac{2}{\pi}} \frac{\sin(\alpha)}{\alpha}$$