November 21

Fourier series

Fourier series:

- Jean-Baptiste Joseph Fourier
- Mathematical tool to represent as sums of sine and cosine modes
- Applications widely throughout engineering, signal processing, physics, computer science, ...

Introduction

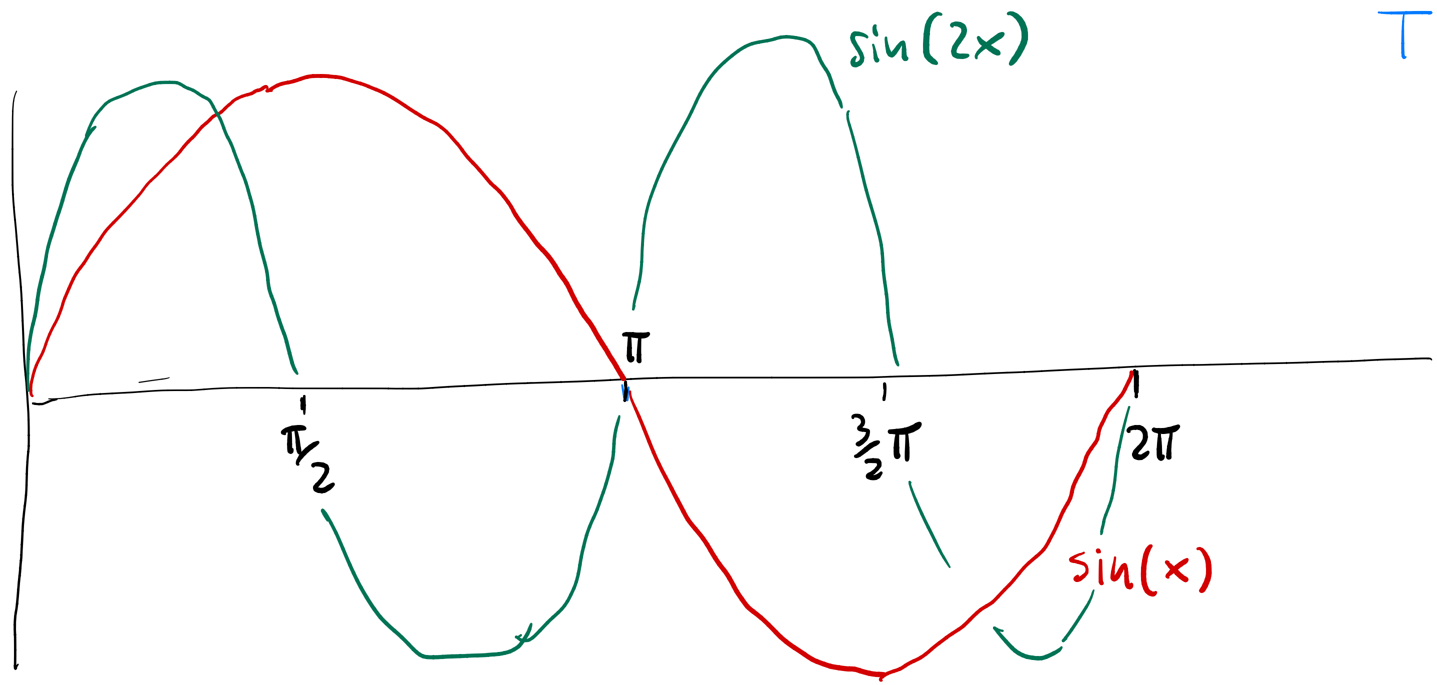
Given a period $T > 0$, for any non-negative integer n we have

$$\cos\left(\frac{2\pi n x}{T}\right) \qquad \sin\left(\frac{2\pi n x}{T}\right)$$

n-th cosine mode with period T

n-th sine mode with period T

$T = 2\pi$



Suppose that N is a non-negative integer. Consider

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)$$

We call this a <u>partial</u> Fourier series

- We call $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 1}$ the coefficients of the sum
- T is the period of this partial Fourier series
- $a_0/2$ is the average of f over $[0, T]$.

This <u>partial Fourier series</u> is a finite sum.
More generally, we study Fourier series as infinite sums.

A Fourier series with period T is an infinite sum of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)$$

Technically, the infinite sum is a limit:

$$F(x) = \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)$$

- Again, $a_0/2$ is the integral of f over $[0, T]$
- The Fourier series is the limit (formally) over the partial Fourier series
- It is not immediately evident, whether the Fourier series converges for some given choice of amplitudes $(a_n)_n$ and $(b_n)_n$

Terminology: A Fourier series that contains only cosine / sine modes is called a Fourier cosine / sine series

Basic questions of Fourier series theory:

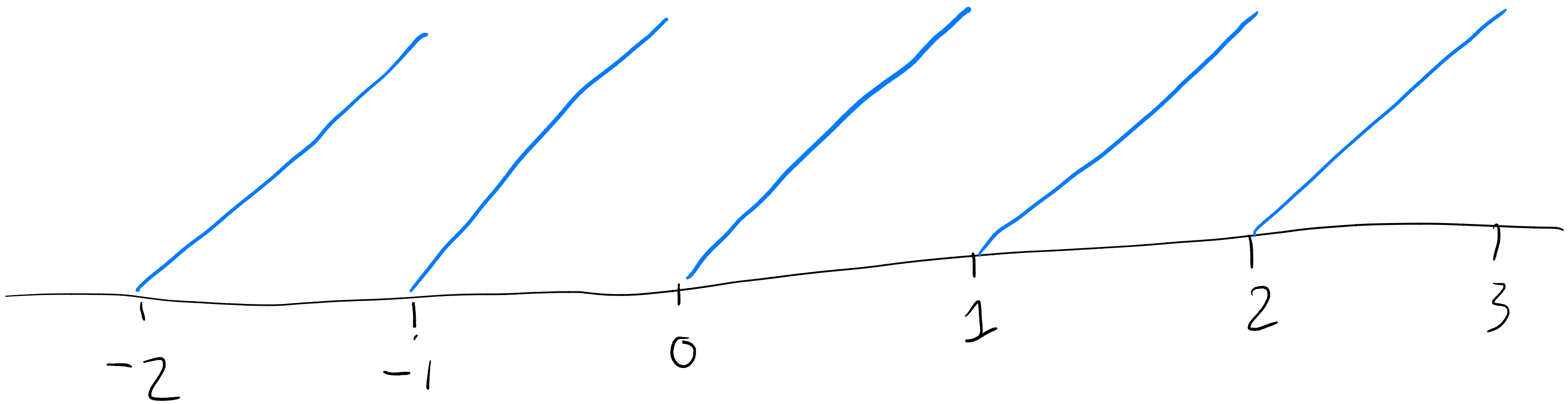
- Given $a_0, a_1, a_2, \dots, b_1, b_2, \dots,$ does the corresponding Fourier series converge?

- Given a periodic function/signal f, can we represent f as a Fourier series and find $a_0, a_1, a_2, \dots, b_1, b_2, \dots$

Examples

1) Suppose $f: \mathbb{R} \to \mathbb{R}$ has period $T = 1$ and satisfies

$$f(x) = x \quad \text{for} \quad 0 \leq x \leq 1$$

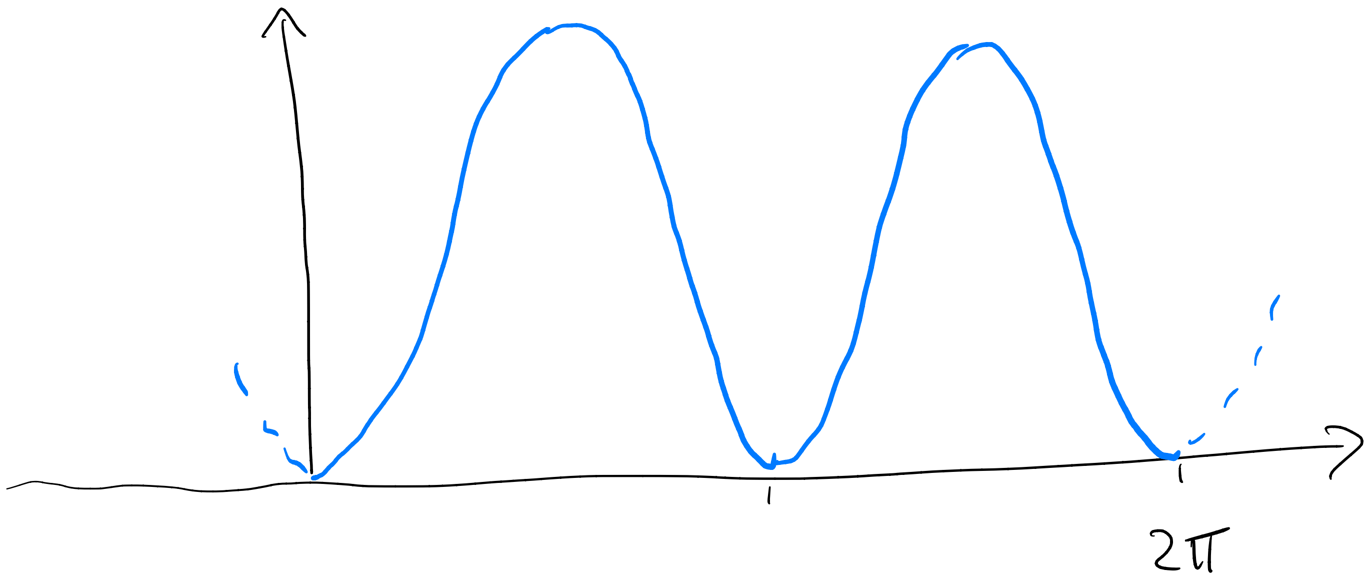


This piecewise linear function is called 'sawtooth wave'

$$f(x) = \underbrace{1/2}_{a_0/2} + \sum_{n=1}^{\infty} \underbrace{\frac{-1}{\pi n}}_{b_n} \sin(2\pi n x)$$

$a_n = 0$

2) Square of sine function, $T = 2\pi$



$$f(x) = \sin^2(x)$$

$$f(x) = \underbrace{\tfrac{1}{2}}_{a_0/2} - \underbrace{\tfrac{1}{2}}_{a_2}\cos(2x)$$

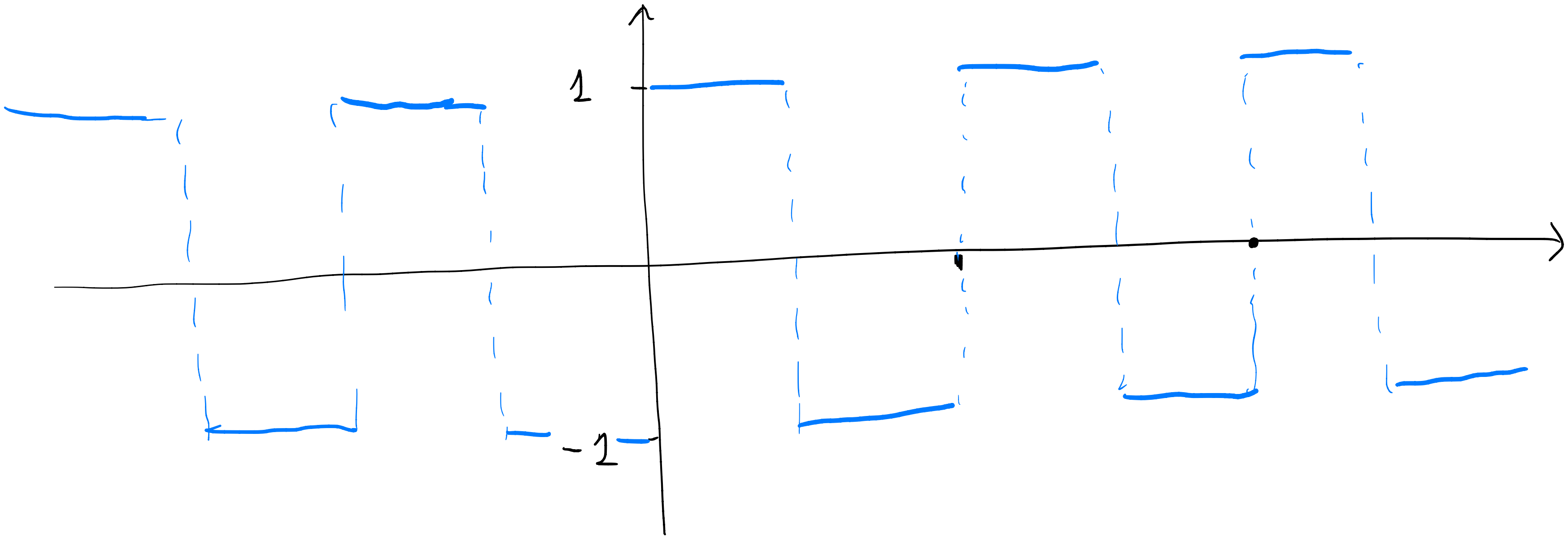
cosine series

$b_n = 0$

$$\cos(2x) = \cos\left(\frac{2\pi n x}{T}\right)$$

3) Square wave with period $T=1$

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \le x < 1 \end{cases}$$



piecewise constant but with discontinuities

Fourier series

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2\pi(2n-1)x)$$

$$= \frac{4}{\pi} \left(\sin(2\pi x) + \frac{1}{3}\sin(2\pi 3x) + \frac{1}{5}\sin(2\pi 5x) + \dots \right)$$

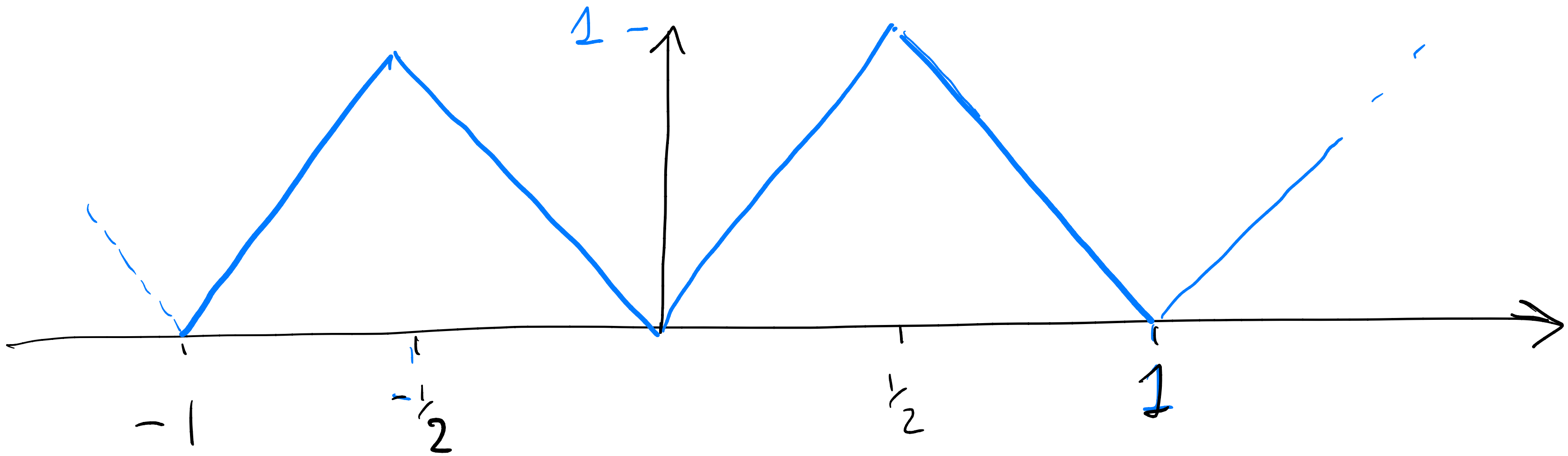
$$= \sum_{n=1}^{\infty} b_n \sin(2\pi n x)$$

where $b_n = \begin{cases} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

At the discontinuities, we observe the Gibb's phenomenon
(that is, overshoot / undershoot)

4) Triangle wave with period 1

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2} \\ 2(1-x) & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$



Fourier series

$$f(x) = \underbrace{\frac{1}{2}}_{a_0/2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \underbrace{\frac{-4}{\pi^2 n^2}}_{\substack{a_n \\ n \text{ odd}}} \cos(2\pi n x)$$

cosine series

$$a_0 = 1 \qquad a_n = \begin{cases} -\frac{4}{\pi^2 n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

<u>Remark</u>: Typically, the Fourier series of a function converges very fast if the function is <u>continuous</u> but if the function has discontinuities, then the Gibb's phenomenon takes place.

5) Dirac comb / pulse train

$$\Delta_1 = \sum_{k=-\infty}^{k=\infty} \delta_k = \sum_{k \in \mathbb{Z}} \delta_k$$

This is a "periodic" distribution with period $T = 1$

$$\langle \Delta_1, \varphi \rangle = \sum_{k \in \mathbb{Z}} \varphi(k)$$

Fourier series:

$$\Delta_1 = 1 + \sum_{n=1}^{\infty} 2 \cos(2\pi n x)$$

This equation is to be interpreted as follows

$$\langle \Delta_1, \varphi \rangle = \int_{-\infty}^{\infty} \varphi(x)\, dx + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} 2 \cos(2\pi n x)\, \varphi(x)\, dx$$

Remark: in some sense, we have got a very pronounced Gibb's phenomenon in the Fourier series of the Dirac comb.

Computing Coefficients of Fourier series

Suppose $f: \mathbb{R} \to \mathbb{R}$ has period T:

$$f(x+T) = f(x)$$

We would like to express f as a Fourier series

We first review facts about the Fourier modes

Recall the following about sine and cosine functions

When $n = 0$, then

$$\int_0^T \cos\left(\frac{2\pi \cdot 0 \cdot x}{T}\right)^2 dx = \int_0^T 1 \, dx = T$$

When $n \geq 1$, then

$$\int_0^T \cos\left(\frac{2\pi n x}{T}\right)^2 dx = \int_0^T \frac{1}{2} + \frac{1}{2}\cos\left(\frac{4\pi n x}{T}\right) dx = T/2$$

$$\int_0^T \sin\left(\frac{2\pi n x}{T}\right)^2 dx = \int_0^T \frac{1}{2} - \frac{1}{2}\cos\left(\frac{4\pi n x}{T}\right) dx = T/2$$

More generally, we want to compute

$$\int_0^T \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \left(\frac{2\pi m x}{T}\right) \cdot \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \left(\frac{2\pi n x}{T}\right) dx$$

We have already computed the case $m = n$ and both are either sine or cosine modes. We want to study $m \neq n$ and mix sine / cosine.

We remember the angle sum formulas:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

Observation 1 Add third and fourth angle sum formula

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos(\alpha)\cos(\beta)$$

If $m, n \geq 0$ with $m \neq n$

$$\int_0^T \cos\left(\frac{2\pi m x}{T}\right)\cos\left(\frac{2\pi n x}{T}\right)$$

$$= \frac{1}{2}\int_0^T \cos\left(\frac{2\pi(m+n)x}{T}\right) + \cos\left(\frac{2\pi(m-n)x}{T}\right) dx$$

since $m+n \neq 0$ and $m-n \neq 0$

$$= 0$$

Observation 2: (subtract third from fourth angle sum formula)

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin(\alpha)\sin(\beta)$$

If $m, n \geq 0$ with $m \neq n$, then

$$\int_0^T \sin\left(\frac{2\pi m x}{T}\right)\sin\left(\frac{2\pi n x}{T}\right)$$

$$= \frac{1}{2}\int_0^T \cos\left(\frac{2\pi(m-n)x}{T}\right) - \cos\left(\frac{2\pi(m+n)x}{T}\right)dx$$

$$= 0$$

Observation 3 : (add first and second angle sum formula)

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta)$$

If $m, n \geq 0$ with $m \neq n$

$$\int_0^T \sin\left(\frac{2\pi m x}{T}\right)\cos\left(\frac{2\pi n x}{T}\right) dx$$

$$= \frac{1}{2}\int_0^T \sin\left(\frac{2\pi(m+n)x}{T}\right) + \sin\left(\frac{2\pi(m-n)x}{T}\right) dx$$

$$= 0$$

To summarize, the cosine and sine modes satisfy the orthogonality relations:

$$\int_0^T \sin\left(\frac{2\pi m x}{T}\right)\cos\left(\frac{2\pi n x}{T}\right) dx = 0$$

$$\int_0^T \cos\left(\frac{2\pi m x}{T}\right)\cos\left(\frac{2\pi n x}{T}\right) dx = \delta_{m,n}\, T/2$$

$$\int_0^T \sin\left(\frac{2\pi m x}{T}\right)\sin\left(\frac{2\pi n x}{T}\right) dx = \delta_{m,n}\, T/2$$

where $m, n \geq 0$ are non-negative integers, not both zero, and we have used the Kronecker delta

$$\delta_{m,n} := \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Remark: The Fourier sine and cosine modes are an orthogonal system with respect to the L^2 pairing

$$\langle f, g \rangle_{L^2} := \int_0^T f(x)\, g(x)\, dx$$

Analogous to how to how the unit vectors $e_1, e_2, \dots, e_n$ are an orthogonal system to the Euclidean product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

In some sense, we generalize orthogonality from finite to infinite dimensions

We use the orthogonality condition to define the Fourier coefficients of a function $f(x)$ with period T.

Suppose $f: \mathbb{R} \to \mathbb{R}$ with period T and integrable over $[0, T]$.

If f can be written as a Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)$$

what happens when we integrate f against a sine/cosine mode?

Let $m \geq 0$, and consider

$$\int_0^T f(x) \cos\left(\frac{2\pi m x}{T}\right) dx$$

$$= \int_0^T \frac{a_0}{2} \cos\left(\frac{2\pi m x}{T}\right) dx + \sum_{n=1}^{\infty} a_n \int_0^T \cos\left(\frac{2\pi n x}{T}\right) \cos\left(\frac{2\pi m x}{T}\right) dx + b_n \int_0^T \sin\left(\frac{2\pi n x}{T}\right) \cos\left(\frac{2\pi m x}{T}\right) dx$$

If $m=0$, this equals

$$\int_0^T \frac{a_0}{2}\,dx = a_0 \, T/2$$

If $m>0$, then this equals

$$\int_0^T a_m \cos\left(\frac{2\pi m x}{T}\right)^2 dx = a_m \, T/2$$

Lastly, if $m > 0$ and we integrate against the m-th sine mode

$$\int_0^T f(x) \sin\left(\frac{2\pi m x}{T}\right) dx$$

$$= \underbrace{\frac{a_0}{2}\int_0^T \sin\left(\frac{2\pi m x}{T}\right) dx}_{=0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_0^T \cos\left(\frac{2\pi n x}{T}\right) \sin\left(\frac{2\pi m x}{T}\right) dx}_{=0} + b_n \underbrace{\int_0^T \sin\left(\frac{2\pi n x}{T}\right) \sin\left(\frac{2\pi m x}{T}\right) dx}$$

$$= b_m \int_0^T \sin\left(\frac{2\pi m x}{T}\right)^2 dx = b_m \, T/2$$

We thus see that integration against sine / cosine modes "extracts" the coefficients of the Fourier series.

$$a_0 = \frac{2}{T}\int_0^T f(x)\,dx, \qquad a_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi n x}{T}\right), \qquad b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi n x}{T}\right)dx$$

<u>Idea</u>: The argument above has assumed that f already can be written as a Fourier series. Taking this as a point of inspiration,

given any $f: \mathbb{R} \to \mathbb{R}$ with period T, we define its <u>Fourier coefficients</u> $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ using those integrals

We can then define the Fourier series of f using those coefficients.
However, the Fourier series might not always converge, and even if it converges, the limit might be different from f.

We introduce some terminology / notation

Let $f : \mathbb{R} \to \mathbb{R}$ with period T and let $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ be the Fourier coefficients, computed by the integral formula above

Fourier series
$$F f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)$$

Partial Fourier series
$$F_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)$$

Fourier cosine series
$$F_c f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right)$$

Fourier sine series
$$F_s f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{T}\right)$$

Examples

1) Saw tooth wave $f: \mathbb{R} \to \mathbb{R}$ with period $T = 1$ and

$$f(x) = x \quad \text{for} \quad 0 \leq x < 1$$

We compute

$$a_0 = 2 \int_0^1 f(x)\, dx = 2 \int_0^1 x\, dx = 1$$

$$a_n = \frac{2}{T} \int_0^1 x \cos\left(\frac{2\pi n x}{T}\right) dx = 2 \int_0^1 x \cos(2\pi n x)\, dx$$

$$= 2 \left(\underbrace{\left[x \frac{\sin(2\pi n x)}{2\pi n} \right]_{x=0}^{x=1}}_{=0} - \underbrace{\int_0^1 \frac{\sin(2\pi n x)}{2\pi n}\, dx}_{=0} \right) = 0$$

$$b_n = \frac{2}{T}\int_0^T x \sin\left(\frac{2\pi n x}{T}\right) dx$$

$$= 2\int_0^1 x \sin(2\pi n x)\, dx$$

$$= 2\left(\left[x \frac{-\cos(2\pi n x)}{2\pi n} \right]_{x=0}^{x=1} - \underbrace{\int_0^1 \frac{-\cos(2\pi n x)}{2\pi n}\, dx}_{=0} \right)$$

$$= 2\,\frac{-1}{2\pi n} = \frac{-1}{\pi n}$$