# Analysis III - 203(d)

Winter Semester 2024

## Session Extra: December 29, 2025

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**Exercise 1** *Recall the ReLU function*

$$\mathrm{ReLU}(x) = \max(0, x). \tag{1}$$

*A shallow neural network with one output neuron, $m$ internal neurons, and $n$ input neurons has the general form*

$$f(x_1, \dots, x_n) = \sum_{k=1}^{m} \mathrm{ReLU}\left( \sum_{i=1}^{n} A_{ki} x_i + b_k \right) \tag{2}$$

*where $A_{ki}$ are weights and where $b_k$ is a shift parameter.*

- *Most training algorithms require the gradient of this network. Compute the derivatives $\partial_i f$*
- *There is interest in training algorithms that use the Hessian matrix. Compute the partial derivatives $\partial^2_{ij} f$.*

**Exercise 2** *In standard models of elasticity, a long straight beam of elastic material, such as wood or metal, can be modeled as a one-dimensional interval. When it is subject to an outside force $f$, such as gravity, than the deformation from the base is modeled by the beam equation*

$$u''''(x) = f(x) \tag{3}$$

*Here, the fourth derivative $u''''$ can be interpreted as the curvature of a curvature and $f$ describes the direction (upwards, downwards) and magnitude of the force.*
*So-called non-local interactions are modeled via a convolutional term $k \star u$, where $k$ indicates how parts of a beam are influenced by neighboring parts. With that in mind, we consider a generalized beam equation*

$$u''''(x) + cu(x) + (k \star u)(x) = f(x). \tag{4}$$

*This is a so-called integro-differential equation.*
*Compute the Fourier transform of this differential equation for general source terms, and write it down for the particular example*

$$k(x) = e^{-|y|}, \quad f(x) = e^{-y^2}. \tag{5}$$

*You are not expected to solve this equation.*

**Exercise 3** *Solve the integro-differential equation*

$$9u(x) + 2\int_{-\infty}^{+\infty} \big(u''(t) - 4u(t)\big)\, e^{-2|x-t|} dt = \frac{1}{x^2+1}.$$

**Exercise 4** *Solve the integral equation*

$$u(t) + \lambda \int_{0}^{+\infty} e^{-|\tau|} u(t-\tau)\ d\tau = e^{-|t|}.$$