Analysis III - 203(d)

Winter Semester 2024

Session 4: October 3, 2024

Exercise 1 *Compute the line integral of the vector field $\vec{F}$ along the curve γ, where*

$$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto (x_2, 0), \qquad \gamma : [0, 2\pi] \to \mathbb{R}^2, \quad t \mapsto (\cos(t), \sin(t))$$

Compute the line integral of the vector field $\vec{G}$ along the curve δ, where

$$\vec{G}: \mathbb{R}^3 \to \mathbb{R}^3, \quad (x_1, x_2, x_3) \mapsto (2, 3, -1), \qquad \delta : [0, 4] \to \mathbb{R}^2, \quad t \mapsto (t^2, \cos(t), e^t)$$

Exercise 2 *Compute the line integral of the vector field $\vec{F}$ along the curve γ, where*

$$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto (e^{x_1 x_2} x_2, e^{x_1 x_2} x_1)$$

and

$$\gamma : [0, 1] \to \mathbb{R}^2, \quad t \mapsto \left(\arctan(\cos(\pi t)^2 - \sin(\pi t)^2), 1 + \sqrt[2]{1 + \arctan(t)}\right)$$

Exercise 3 *What is the length of the graph of the function $g(x) = x^{\frac{3}{2}}$ over the interval $[0, 2]$? Simplify as much as reasonable.*

Exercise 4 *Take a look at the functions*

$$f(x_1, x_2) = 1, \quad g(x_1, x_2) = x_2.$$

We have the following curves:

$$\begin{aligned}
&\alpha : [0, 1] \to \mathbb{R}^2, \quad t \mapsto (t, t), \\
&\beta : [0, 2] \to \mathbb{R}^2, \quad t \mapsto \begin{cases} (t, 0) & 0 \le t < 1, \\ (0, t-1) & 1 \le t \le 2, \end{cases} \\
&\gamma : [0, 1] \to \mathbb{R}^2, \quad t \mapsto (t^2, t).
\end{aligned}$$

Compute:

$$\int_A f \, d\ell, \quad \int_B f \, d\ell, \quad \int_A g \, d\ell, \quad \int_B g \, d\ell, \quad \int_\Gamma g \, d\ell.$$

Exercise 5 *We want to verify that the curve integrals do not depend on the direction of the parameterization. Consider the curve parameterizations*

$$\gamma_+ : [0,1] \to \mathbb{R}^2, \quad t \mapsto (t, 1 + \frac{1}{2}t^2), \tag{1}$$

$$\gamma_- : [0,1] \to \mathbb{R}^2, \quad s \mapsto (1-s, 1 + \frac{1}{2}(1-s)^2), \tag{2}$$

- *Given the scalar field*

  $$f(x,y) = x(y - \frac{1}{2}x^2) = xy - \frac{1}{2}x^3,$$

  show that

  $$\int_{\gamma_+} f \; dl = \int_{\gamma_-} f \; dl \tag{3}$$

- *Compute the tangent and unit tangent vectors of each curve, and show that* $\dot{\gamma}_+(t) = -\dot{\gamma}_-(1-t)$*. Show that the curve integrals along* γ_+ *and* γ_- *of the vector field*

  $$F(x,y) = (-y, x) \tag{4}$$

  are the negative of each other.

Exercise 6 *Consider the open cube*

$$\Omega = (0,1)^2 = \left\{(x,y) \in \mathbb{R}^2 : 0 < x, y < 1\right\}.$$

- *Show that* Ω *is convex.*
- *Let* $z \in \Omega$*. Show that* Ω *is star-shaped with respect to* z*.*
- *Show that* Ω *is simply-connected.*

For the last part, show that any two continuous curves with the same endpoints are homotopic relative to endpoints.