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Fourier transform (cont.)

$$\hat{f}(\alpha) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\alpha t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} e^{-i\alpha t}\, dt = \frac{1}{\sqrt{2\pi}} \left[ \frac{e^{-i\alpha t}}{-\alpha i} \right]_{t=-1}^{t=+1}$$

$$= \frac{e^{i\alpha} - e^{-i\alpha}}{\sqrt{2\pi} \cdot \alpha i}$$

We can simplify this further:

$$= \frac{1}{\alpha\sqrt{2\pi}} \frac{e^{i\alpha} - e^{-i\alpha}}{i} = \frac{2}{\alpha\sqrt{2\pi}} \frac{e^{i\alpha} - e^{-i\alpha}}{2i} = \sqrt{\frac{2}{\pi}} \frac{\sin(\alpha)}{\alpha}$$

Here, we have used

$$\frac{e^{i\alpha} - e^{-i\alpha}}{2i} = \frac{\cos(\alpha) + \sin(\alpha) i - \cos(-\alpha) - \sin(-\alpha) i}{2i}$$

$$= \frac{\cancel{\cos(\alpha)} + \sin(\alpha) i - \cancel{\cos(\alpha)} + \sin(\alpha) i}{2i}$$

$$= \frac{2\sin(\alpha) i}{2i} = \sin(\alpha)$$

In computing the integral, we used the antiderivative of a complex-valued function. Formally, we can also compute

$$\int_{-1}^{1} e^{-i\alpha t}\,dt = \int_{-1}^{+1} \cos(-\alpha t) + \sin(-\alpha t)\,i \;\, dt$$

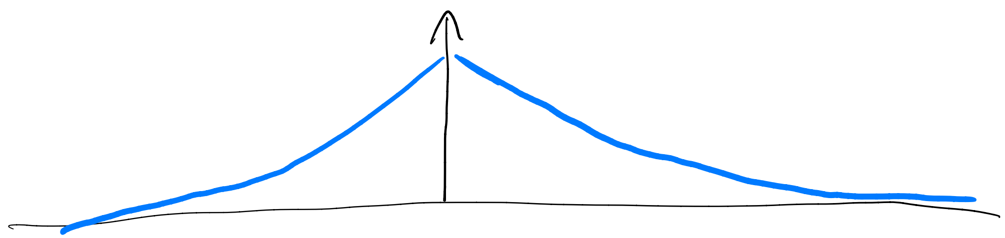
$$= \left[ \frac{\sin(-\alpha t)}{-\alpha} - \frac{\cos(-\alpha t)}{-\alpha}\,i \right]_{t=-1}^{t=+1}$$

$$= \left[ \frac{\cos(-\alpha t)}{-\alpha i} + \frac{\sin(-\alpha t)}{-\alpha i}\,i \right]_{t=-1}^{t=+1} = \left[ \frac{e^{-i\alpha t}}{-\alpha i} \right]_{t=-1}^{t=+1}$$

Both methods lead to the same result.

**Example 2** Consider the signal $f : \mathbb{R} \to \mathbb{R}$

$$f(t) = e^{-|t|}$$



We compute the Fourier transform

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\alpha t} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-|t| - i\alpha t} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t - i\alpha t} \, dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{t - i\alpha t} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{t(-1-i\alpha)} dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{t(1-i\alpha)} dt$$

We compute both seperately. On the one hand,

$$\int_0^\infty e^{t(-1-i\alpha)} dt = \left[ \frac{e^{t(-1-i\alpha)}}{(-1-i\alpha)} \right]_{t=0}^{t=\infty}$$

$$= \underbrace{\lim_{t\to\infty} \frac{e^{-t} e^{-i\alpha t}}{(-1-i\alpha)}}_{\longrightarrow 0} - \frac{e^0}{(-1-i\alpha)} = \frac{1}{1+i\alpha}$$

Analogously, we compute

$$\int_{-\infty}^0 e^{t(1-i\alpha)} dt = \left[ \frac{e^{t(1-i\alpha)}}{1-i\alpha} \right]_{t=-\infty}^{t=0}$$

$$= \frac{1}{1-i\alpha} - \underbrace{\lim_{t \to -\infty} \frac{e^{t} e^{-i\alpha t}}{1-i\alpha}}_{\to 0} = \frac{1}{1-i\alpha}$$

We thus combine:

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{1+i\alpha} + \frac{1}{1-i\alpha} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(1-i\alpha) + (1+i\alpha)}{(1+\alpha i)(1-\alpha i)} = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$$

# Dirichlet Theorem and Inverse Fourier transform

Recall the Dirichlet theorem for the Fourier series: the original signal can be rebuilt from the Fourier series

Analogue for the Fourier transform: the original signal $f(t)$ can be reconstructed from $\hat{f}(\omega)$

We use the inverse Fourier transform

Inverse Fourier transform

$$\mathcal{F}^{-1}(\hat{f})(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha)\, e^{i\alpha t}\, d\alpha$$

<u>Remarks</u>:

- Only difference between $\mathcal{F}$ and $\mathcal{F}^{-1}$ is the sign in the exponent
- As with the Fourier transform, different authors define the inverse Fourier transform in different ways, and some authors will switch the definition of the two.

The Fourier transform and the inverse Fourier transform are connected by the

# Dirichlet theorem

Suppose that $f: \mathbb{R} \to \mathbb{R}$ is a continuous function.
If the Fourier transform exists for all frequencies $\alpha$, then for all $t \in \mathbb{R}$:

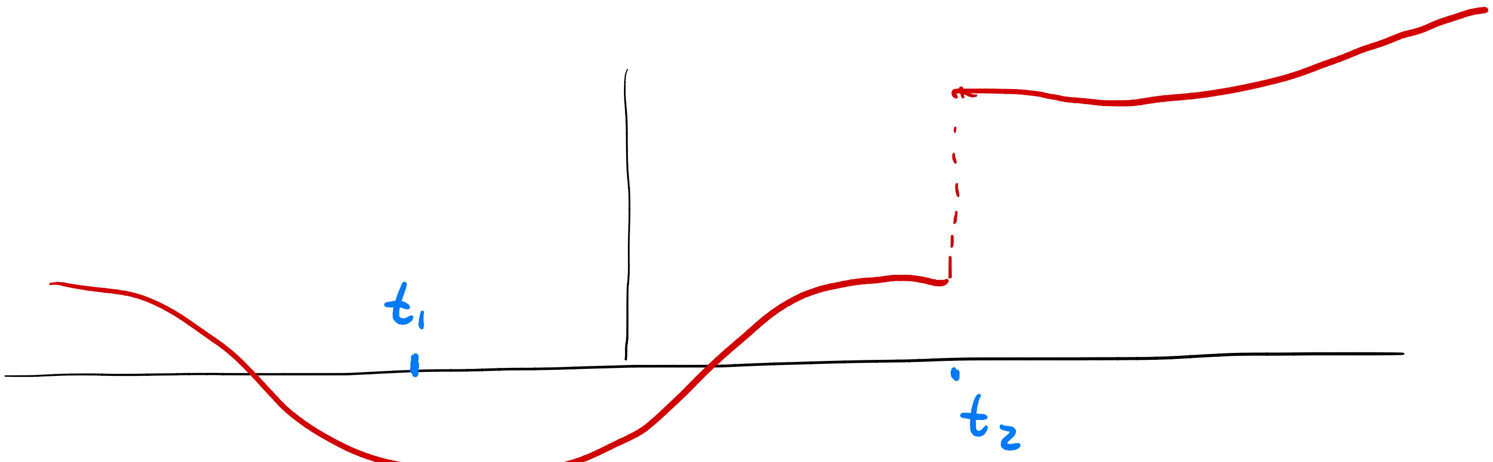
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha)\, e^{i\alpha t}\, d\alpha$$

More generally, if $f$ piecewise continuous, then for all $t$:

$$\frac{1}{2} \lim_{h \to 0} f(t+h) + f(t-h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha)\, e^{i\alpha t}\, d\alpha$$

Remarks: - completely analogous to the Dirichlet theorem for the Fourier series.

- if $f$ is continuous at $t_1$ then the inverse FT reconstructs $f(t)$
- if instead $f$ has a jump at $t_1$ then the inverse FT reconstructs the average of the values of $f$ to the left and the right of the jump.




$$f(t_1) = (\mathcal{F}^{-1}\mathcal{F} f)(t_1) \quad \checkmark$$

$$\lim_{h \to 0} \frac{f(t_2+h) + f(t_2-h)}{2} = (\mathcal{F}^{-1}\mathcal{F} f)(t_2)$$

# Important properties of the FT

<u>Linearity</u>. If $f, g : \mathbb{R} \to \mathbb{R}$ are signals and $a, b \in \mathbb{R}$, then

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

<u>Interpretation</u>: If we amplify the signal $f$ by a factor $a$, then the FT is amplified by the same factor

If we have the superposition of two signals, then its FT is the superposition of the two FTs.

Derivatives: If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable signal, then

$$\mathcal{F}(f')(\alpha) = i\alpha \, \mathcal{F}(f)(\alpha)$$

Interpretation: Differentiation in the time representation of a signal corresponds to multiplication by $i\alpha$ in the frequency representation of the signal.

This is a powerful tool for differential equations

If $f : \mathbb{R} \to \mathbb{R}$ has derivatives up to order $n$, then

$$\mathcal{F}(f^{(n)})(\alpha) = (i\alpha)^n \, \mathcal{F}(f)(\alpha)$$

**Plancherel:** If $f : \mathbb{R} \to \mathbb{R}$ is a signal, then

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{f}(\alpha)|^2 \, d\alpha$$

Interpretation: the original signal $f(t)$ and its Fourier transform $\hat{f}(\alpha)$ have the same energy.

<u>Modulation:</u> If $f : \mathbb{R} \to \mathbb{R}$ is a signal and $a, b \in \mathbb{R}$ with $a \neq 0$ then

$$g(t) = e^{-ibt} f(at)$$

has the Fourier transform

$$\hat{g}(\alpha) = \frac{1}{|a|} \hat{f}\left( \frac{\alpha + b}{a} \right)$$

<u>Interpretation:</u>

If we stretch a signal by $a \neq 0$, then the FT is squeezed

If we modulate a signal by $e^{-ibt}$, then the FT is shifted.

# Fourier transform table

In what follows, let $\omega > 0$

$$f(t) = \frac{1}{t^2 + \omega^2} \iff \hat{f}(\alpha) = \sqrt{\frac{\pi}{2}}\, \frac{e^{-\omega|\alpha|}}{\omega}$$

$$f(t) = \frac{e^{-\omega|t|}}{\omega} \iff \hat{f}(\alpha) = \sqrt{\frac{2}{\pi}}\, \frac{1}{\omega^2 + \alpha^2}$$

$$f(t) = \frac{\sin(\omega t)}{t} \iff \hat{f}(\alpha) = \begin{cases} \sqrt{\pi/2} & |\alpha| < \omega \\ 0 & \text{otherwise} \end{cases}$$

$$f(t) = \begin{cases} 1 & \text{if } |t| < b \\ 0 & \text{otherwise} \end{cases} \iff \hat{f}(\alpha) = \sqrt{\frac{2}{\pi}}\, \frac{\sin(b\alpha)}{\alpha}$$

$$f(t) = e^{-\omega^2 t^2} \iff \hat{f}(\alpha) = \frac{1}{\sqrt{2}\,\omega} e^{-\frac{\alpha^2}{4\omega^2}}$$

$$f(t) = te^{-\omega^2 t^2} \iff \hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2}\,\omega^3} e^{-\frac{\alpha^2}{4\omega^2}}$$

$$f(t) = \frac{4t^2}{(\omega^2+t^2)^2} \iff \hat{f}(\alpha) = \sqrt{2\pi}\left(\frac{1}{\omega} - |\alpha|\right) e^{-\omega|\alpha|}$$

$$f(t) = \begin{cases} 1 & b < t < c \\ 0 & \text{otherwise} \end{cases} \iff \hat{f}(\alpha) = \frac{e^{-i\alpha b} - e^{-i\alpha c}}{i\alpha\sqrt{2\pi}}$$

$$f(t) = \begin{cases} e^{-\omega t} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases} \iff \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{\omega + i\alpha}$$

$$f(t) = \begin{cases} e^{-\omega t} & \text{if } b<t<c \\ 0 & \text{otherwise} \end{cases}$$

$$\Longleftrightarrow \quad \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \, \frac{e^{-(\omega+i\alpha)b} - e^{-(\omega+i\alpha)c}}{\omega + i\alpha}$$

$$f(t) = \begin{cases} e^{-i\omega t} & \text{if } b<t<c \\ 0 & \text{otherwise} \end{cases}$$

$$\Longleftrightarrow \quad \hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \, \frac{e^{-i(\omega+\alpha)b} - e^{-i(\omega+\alpha)c}}{\omega + \alpha}$$

Remarks: - Even though we have the FT for signals $f: \mathbb{R} \to \mathbb{R}$, we can use the same formulas for complex-valued signals $f: \mathbb{R} \to \mathbb{C}$

- Any piecewise continuous signal can be written as the sum of its pieces, and therefore the FT of that signal will be the sum of the FT of each piece.

# Applications of Fourier Analysis

Major area of applications: differential equations, in particular: Poisson problem

# Outline of Poisson problem

Suppose $I = (a,b)$ is an open interval

The Poisson problem asks for a twice-differentiable function $u : [a,b] \to \mathbb{R}$ satisfying

$$-\Delta u = f$$

for some function $f : [a,b] \to \mathbb{R}$.

We call $f$ right-hand side or source term / source function

We study <u>Dirichlet boundary conditions</u>

$$u(a) = g_a, \qquad u(b) = g_b$$

for some $g_a, g_b \in \mathbb{R}$.

Reminder of notation

$$-\Delta u(x) = -u''(x) = \partial_x \partial_x u(x) = \partial^2_{xx} u(x)$$

More generally, the right-hand side can be a distribution and the derivatives $u''$ are taken in the sense of distributions.

# Physical background of Poisson problem

Suppose we study the diffusion of heat over some 1D range.
The change of heat will be

$$\partial_t u(x,t) = \partial_x \partial_x u(x,t) + f(x,t)$$

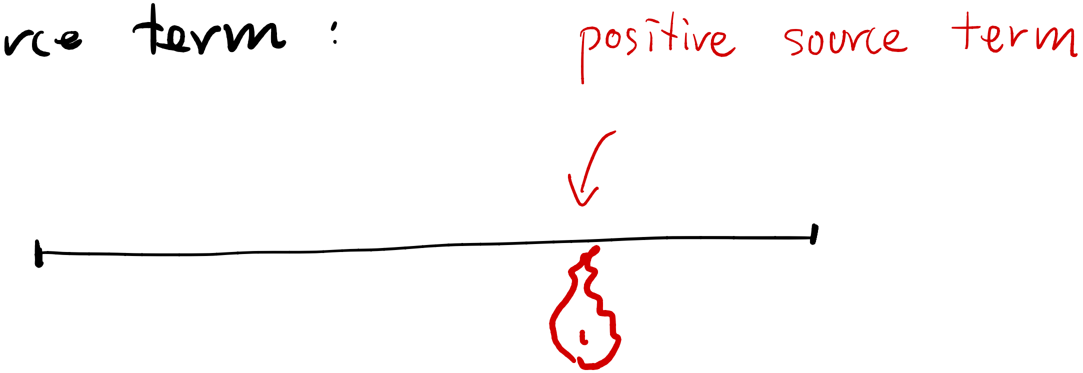
the rate of change of the heat at position-time $(x,t)$

second $x$-derivative of $u$ at position-time $(x,t)$

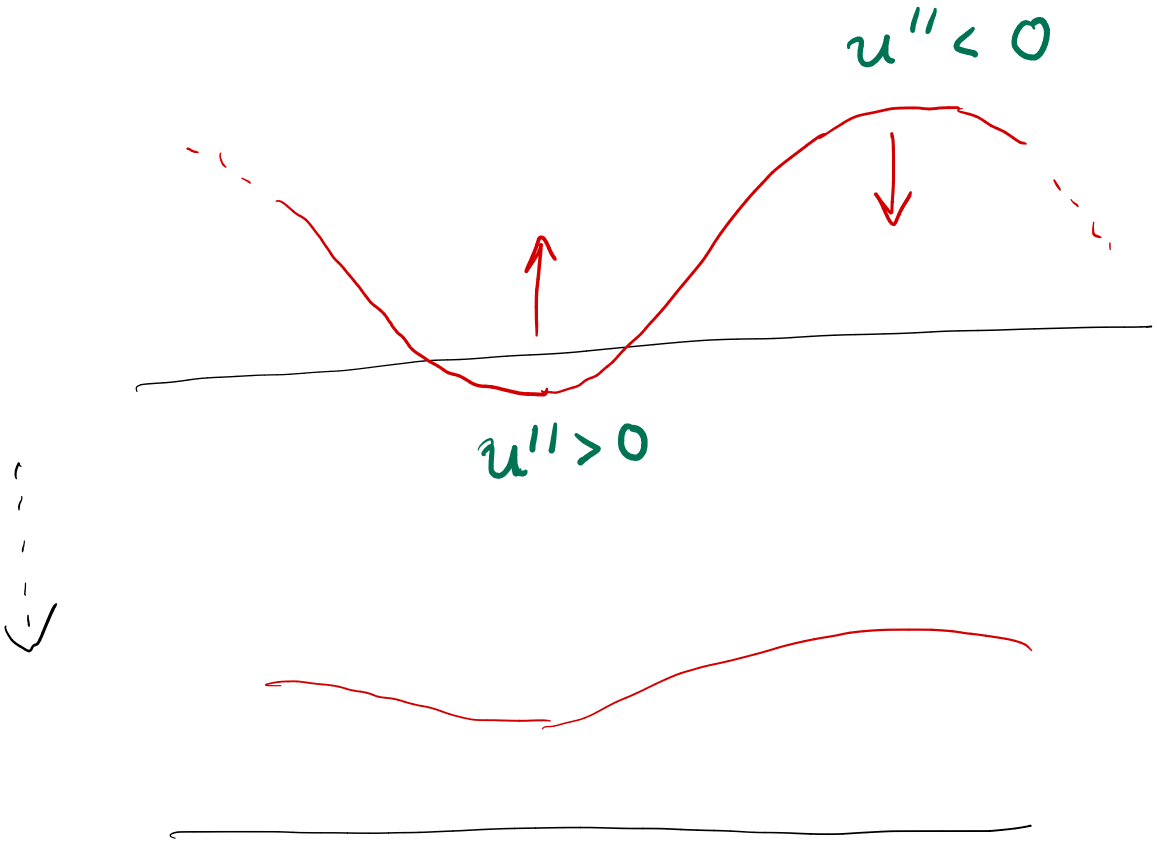
heat production/consumption at position-time $(x,t)$

where $u(x,t)$ is the density of the heat at position $x$ and time $t$.

Source term :




Diffusion: heat fills up sinks and leaves heights




The influence of the diffusion will be proportional to the downward / upward curvature $u''$ of the graph

Suppose that the heat source/sink $f$ does not depend on time but only on the position $x$.

Suppose, we fix the temperature at the endpoints

$$u(a,t) = g_a, \qquad u(b,t) = g_b \qquad t > 0$$

Over time, the heat diffusion will lead to a steady state that does not depend on time. Then, $\partial_t u = 0$

The steady state solution satisfies

$$0 = u'' + f$$

$$u(a) = g_a, \qquad u(b) = g_b$$

Poisson problem with Dirichlet boundary conditions.

We study the Poisson problem with Dirichlet boundary conditions, also called Dirichlet boundary value problem:

Find $u : [a,b] \longrightarrow \mathbb{R}$ such that

$$-\Delta u(x) = f(x)$$

$$u(a) = g_a, \qquad u(b) = g_b$$

Exact solution methods: Green's functions
Fourier series
Fourier transform

Approximate solution methods: finite differences
finite element methods

Superposition principle Suppose we have solutions to the Dirichlet boundary value problems

1st $\begin{cases} -\Delta \tilde{u} = \tilde{f} \\ \tilde{u}(a) = \tilde{g}_a, \quad \tilde{u}(b) = \tilde{g}_b \end{cases}$

2nd $\begin{cases} -\Delta \mathring{u} = \mathring{f} \\ \mathring{u}(a) = \mathring{g}_a, \quad \mathring{u}(b) = \mathring{g}_b \end{cases}$

over the same interval $[a,b]$ but with different data

$\tilde{f}, \tilde{g}_a, \tilde{g}_b \qquad \mathring{f}, \mathring{g}_a, \mathring{g}_b$

Then we have a solution to another Dirichlet boundary value problem:

$$-\Delta\,(\underbrace{\tilde{u} + \mathring{u}}_{u}) = (\underbrace{\tilde{f} + \mathring{f}}_{f})$$

$$(\underbrace{\tilde{u} + \mathring{u}}_{u})(a) = \underbrace{\tilde{g}_a + \mathring{g}_a}_{g_a}, \qquad (\underbrace{\tilde{u} + \mathring{u}}_{u})(b) = \underbrace{\tilde{g}_b + \mathring{g}_b}_{g_b}$$

We see that $u$ satisfies the Dirichlet boundary value problem with data $f$ and $g_a$, $g_b$.