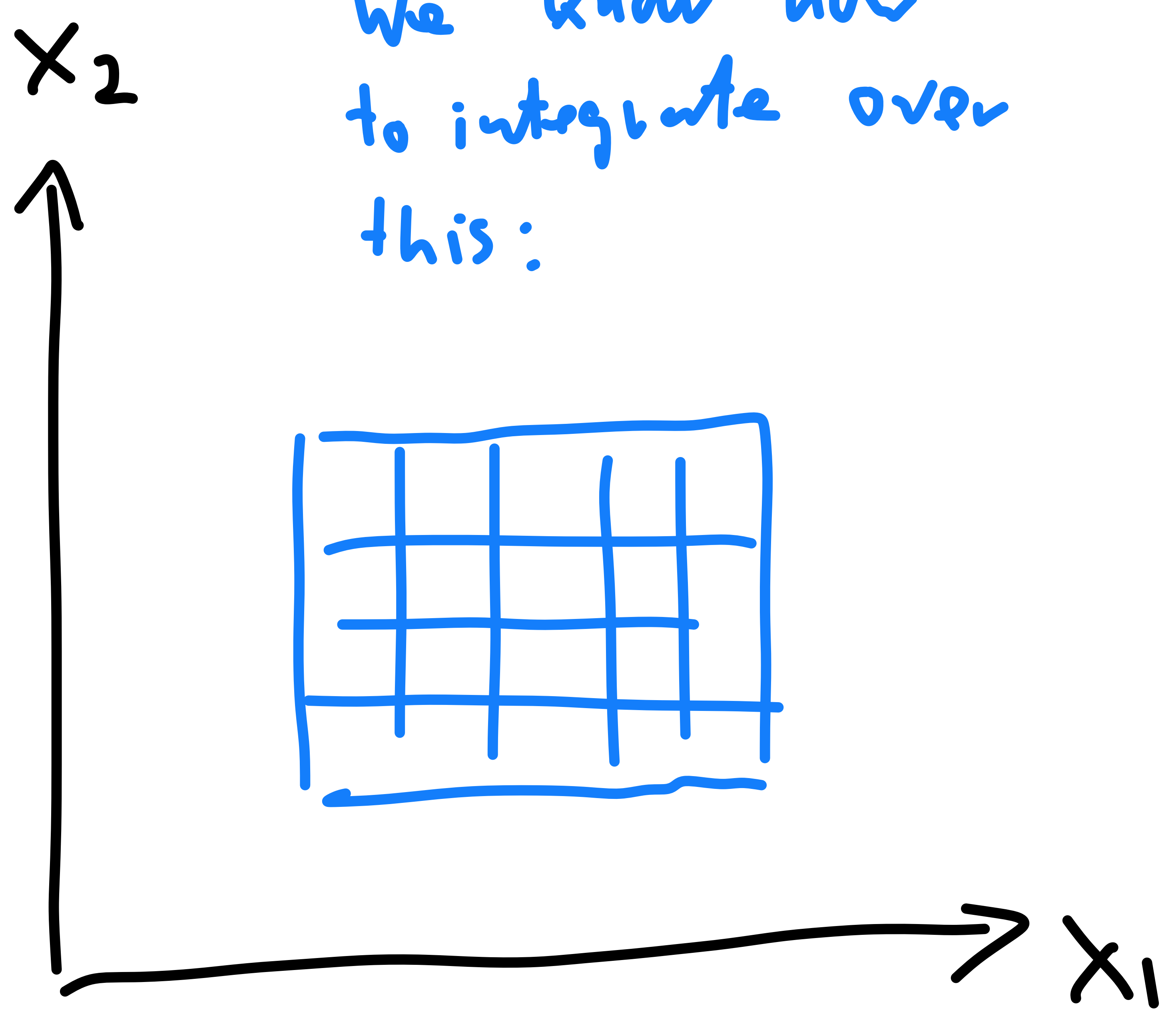
Analysis III - Slides 05

Integrals over surfaces

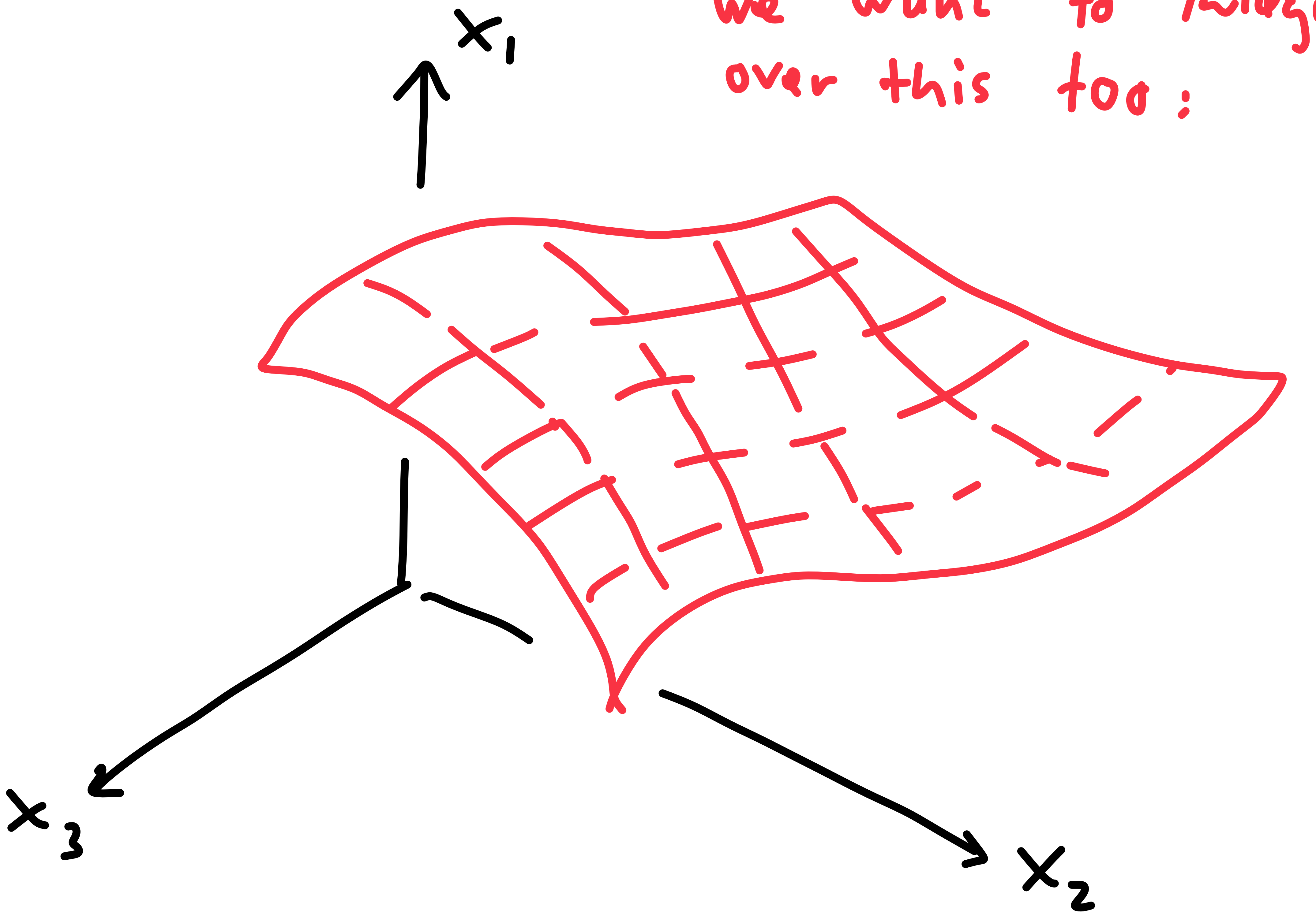
Agenda

Before we continue with vector calculus,
we need to understand <u>surface integrals</u>

We know how to integrate over this:



We want to integrate over this too:



2

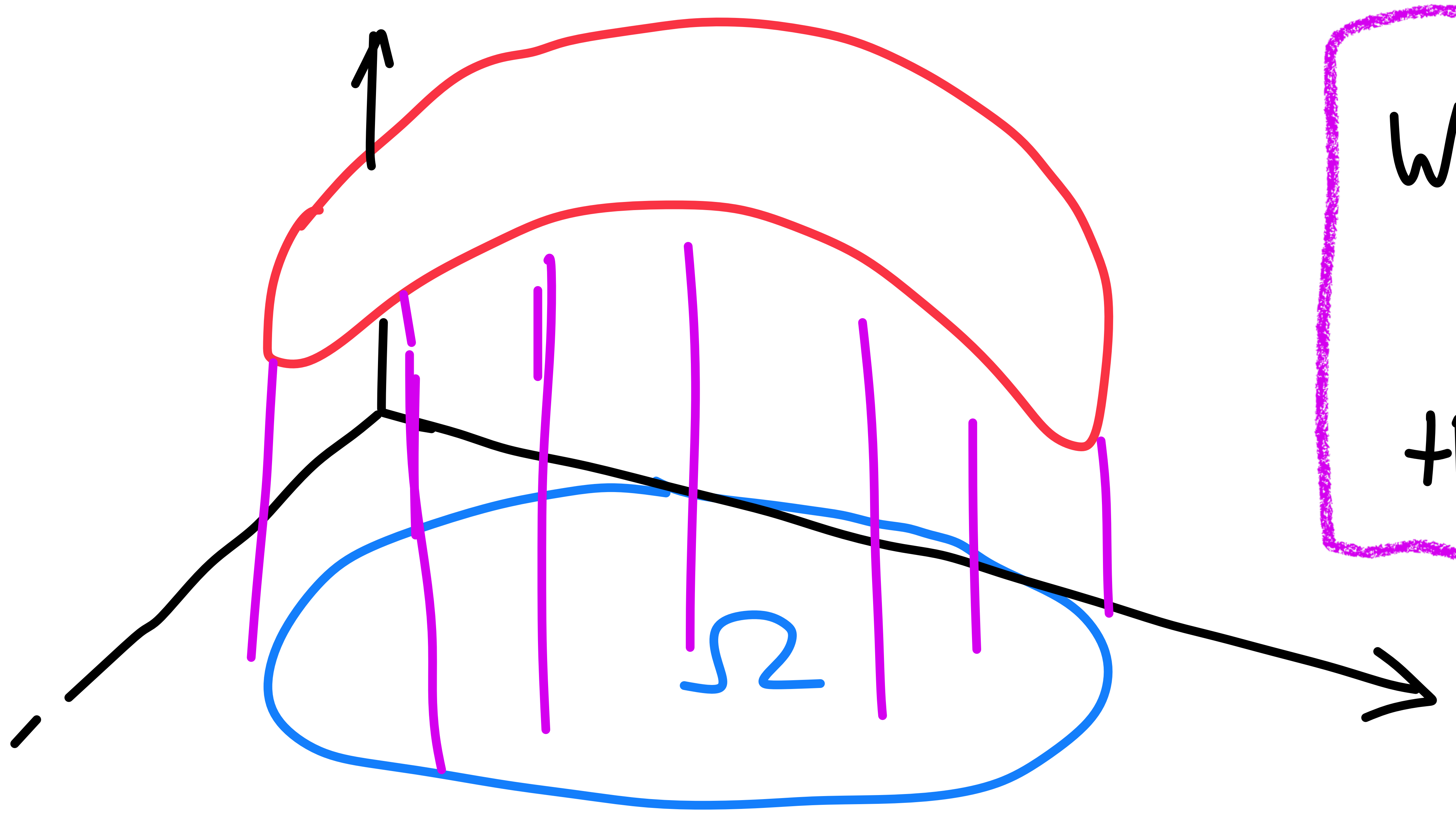
Graph surfaces

Graph surfaces

Let $\Omega \subseteq \mathbb{R}^2$ be a domain and $\phi : \Omega \to \mathbb{R}$ a function.

We study the graph of ϕ over Ω:

$$S := \{ (s, t, \phi(s,t)) \mid (s,t) \in \Omega \}$$



We want to integrate functions

$$f : S \longrightarrow \mathbb{R}$$

that are defined over S.

4

Graph surfaces

It will be easier to work with

$$\Phi : \underset{\mathbb{R}^2}{\underset{\cap}{\Omega}} \longrightarrow \mathbb{R}^3, \qquad (s,t) \longmapsto (s, t, \phi(s,t))$$

5

Graph surfaces

Definition

Let $\Omega \subseteq \mathbb{R}^2$ be a domain and $\phi : \Omega \to \mathbb{R}$ be differentiable.
Let S be the graph of ϕ over Ω.

Let $f : S \longrightarrow \mathbb{R}$ be a scalar field defined over the surface S.
The integral of f over S is:

$$\iint_S f \, d\sigma := \iint_\Omega f(\Phi(s,t)) \cdot \| \partial_s \Phi \times \partial_t \Phi \| \, ds \, dt$$

6

Graph surfaces

With our choice of Φ:

$$\left\| \partial_s \Phi \times \partial_t \Phi \right\| = \left\| \begin{pmatrix} 1 \\ 0 \\ \partial_s \varphi \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \partial_t \varphi \end{pmatrix} \right\| = \left\| \begin{pmatrix} -\partial_s \varphi \\ -\partial_t \varphi \\ 1 \end{pmatrix} \right\|$$

$$= \sqrt{(\partial_s \varphi)^2 + (\partial_t \varphi)^2 + 1}$$

Therefore, for graph surfaces:

$$\iint_S f \, d\sigma = \iint_\Omega f(\Phi(s,t)) \sqrt{(\partial_s \varphi)^2 + (\partial_t \varphi)^2 + 1} \, ds \, dt$$

7

Graph surfaces: example

Suppose $\Omega = [0,1] \times [0,1]$ and $\varphi(s,t) = 2 + 3s + 4t$.

$$\iint_S f \, d\sigma = \iint_\Omega f(s,t,\varphi(s,t)) \sqrt{1 + 3^2 + 4^2} \, ds \, dt$$

For example, if $f(x_1, x_2, x_3) = x_1^2 x_2 x_3$, then

$$\iint_S f \, d\sigma = \iint_\Omega s^2 t \, (2 + 3s + 4t) \sqrt{26} \, ds \, dt$$

$$= \iint_\Omega (2 s^2 t + 3 s^3 t + 4 s^2 t^2) \sqrt{26} \, ds \, dt$$

$$= \dots = \left(\frac{1}{3} + \frac{3}{8} + \frac{4}{9}\right) \sqrt{26} = \frac{83}{72} \sqrt{26}$$

8

Graph surfaces: example

Suppose $\Omega = [0,1] \times [0,1]$ and $\varphi(s,t) = 2 + 3s + 4t$.

$$\iint_S f \, d\sigma = \iint f(s,t,\varphi(s,t)) \sqrt{1+3^2+4^2} \, ds \, dt$$

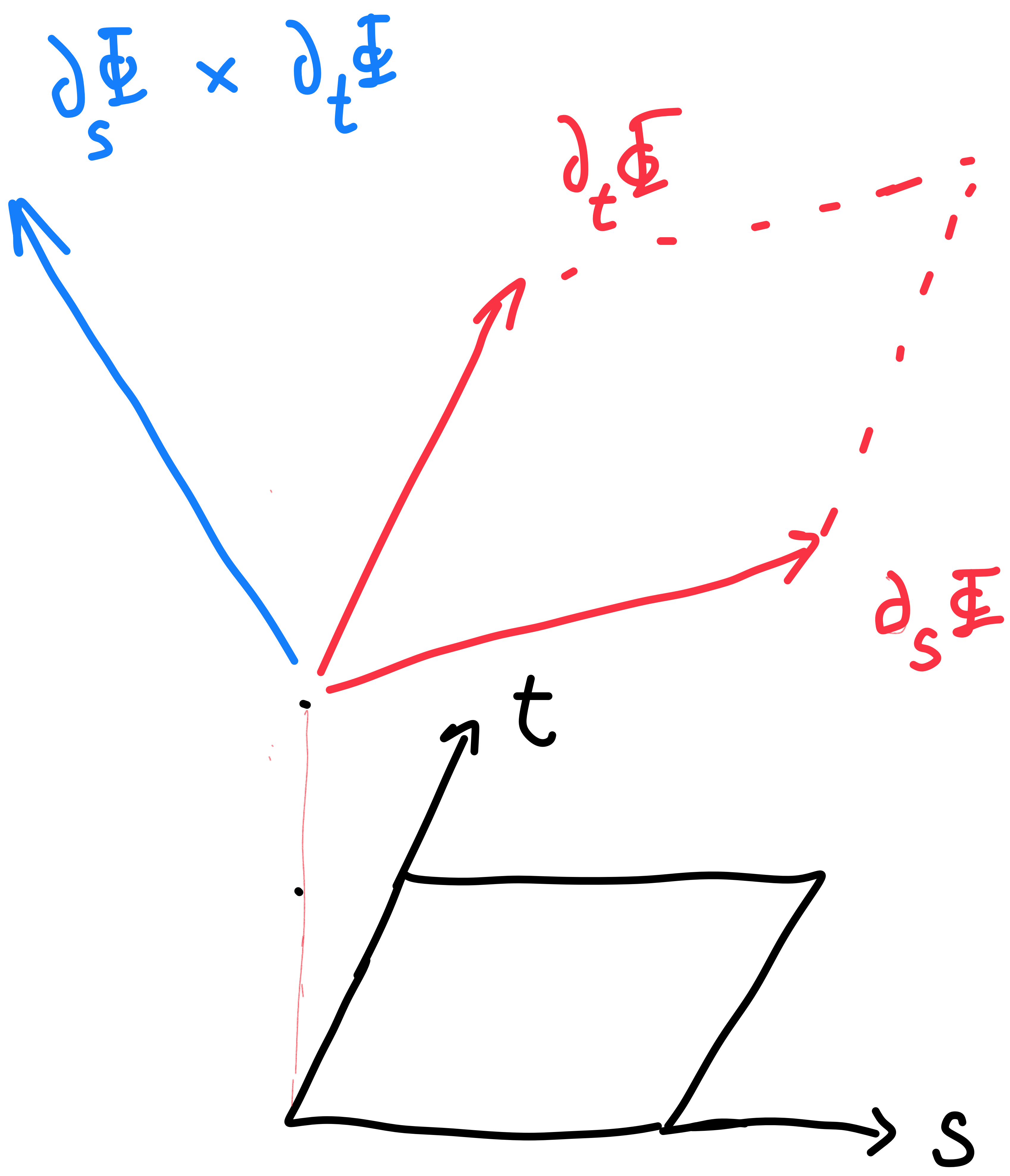
Another example: $f(x_1, x_2, x_3) = 1$. Then

$$\iint_S f \, d\sigma = \iint_S 1 \, d\sigma = \int_0^1 \int_0^1 \sqrt{26} \, dx_1 dx_2 = \sqrt{26}$$

surface area of S

9

Graph surfaces: example



$$\partial_s \Phi = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \qquad \partial_t \Phi = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

$$\partial_s \Phi \times \partial_t \Phi = \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix}$$

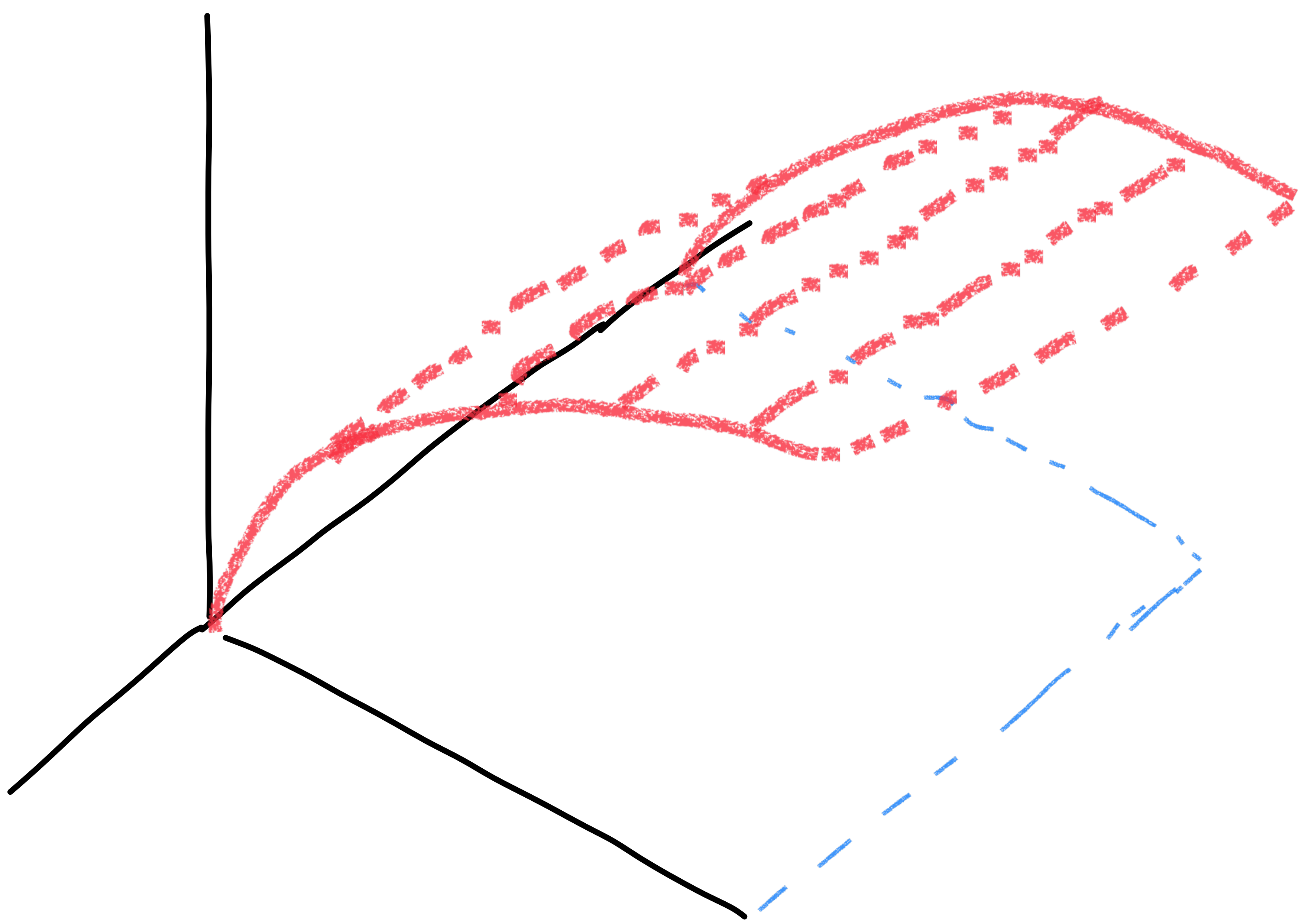
We keep this in mind.

We will revisit this observation later.

10

Graph surfaces: example in one variable

Suppose $\Omega = [0,1] \times [0,1]$ and $\varphi(s,t) = \frac{2}{3} s^{3/2}$



$$S = \{ (s, t, \varphi(s,t)) \in \mathbb{R}^3 \mid 0 \leq s,t \leq 1 \}$$

11

Graph surfaces: example in one variable

For any function $f : S \longrightarrow \mathbb{R}$ we have

$$\iint_S f \, d\sigma = \iint_\Omega f(s, t, \varphi(s,t)) \sqrt{1 + \partial_s \varphi(s,t)^2} \, ds dt$$

$$= \iint_\Omega f(s, t, \varphi(s,t)) \sqrt{1 + s} \, ds dt$$

$$= \int_0^1 \int_0^1 f(s, t, \tfrac{2}{3} s^{3/2}) \sqrt{1 + s} \, ds dt$$

12

Graph surfaces: example in one variable

- For $f(x,y,z) = 1$, we obtain the surface area of the graph

$$\iint_S 1\, d\sigma = \int_0^1 \int_0^1 \sqrt{1+s}\, ds\, dt$$

$$= \int_0^1 \frac{2}{3}(1+s)^{\frac{3}{2}} \Big|_{s=0}^{s=1} dt$$

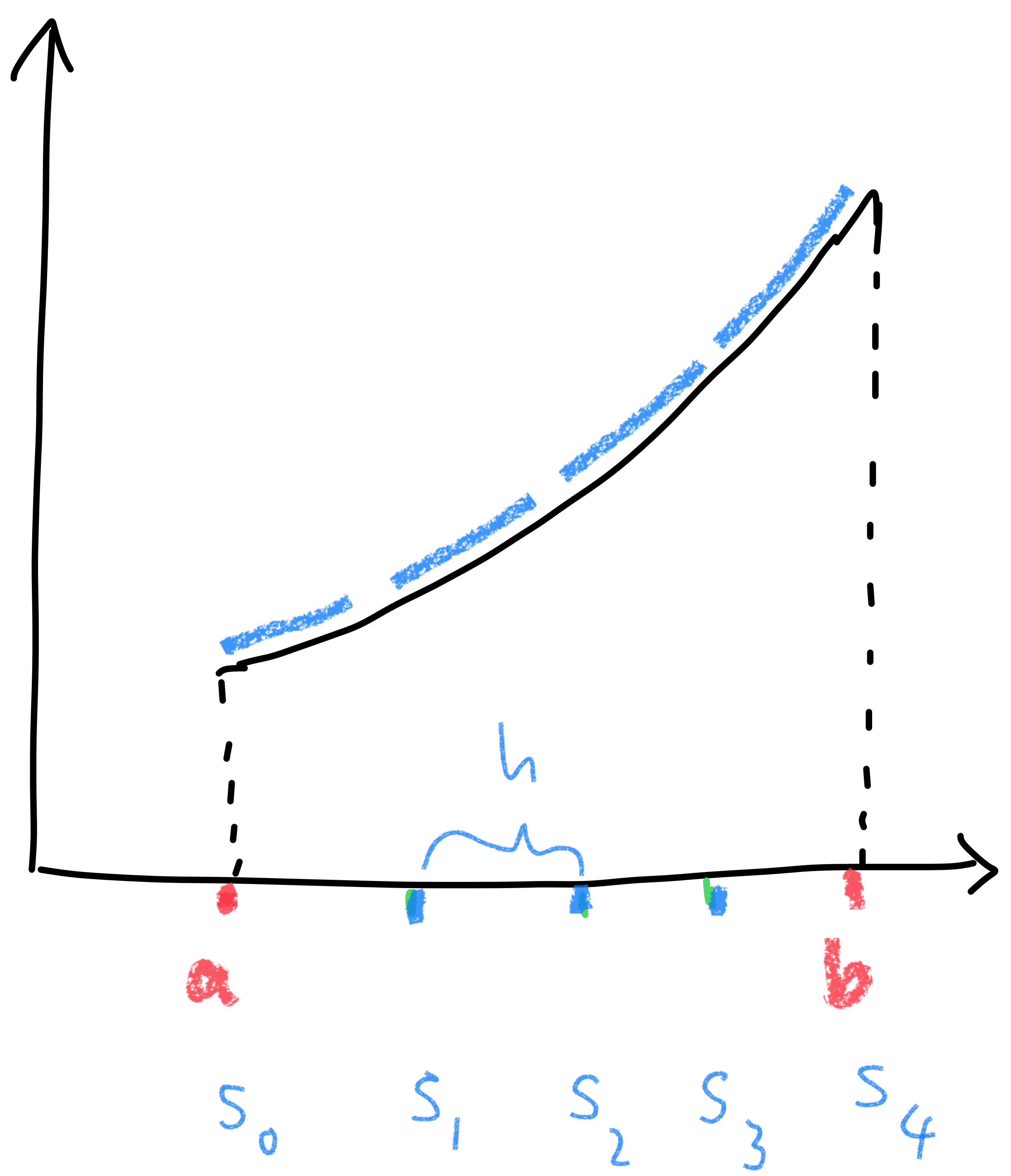
$$= \frac{2}{3}(1+s)^{3/2} \Big|_{s=0}^{s=1} = \frac{2}{3}\left(2^{3/2} - 1\right)$$

13

Graph surfaces: example in one variable

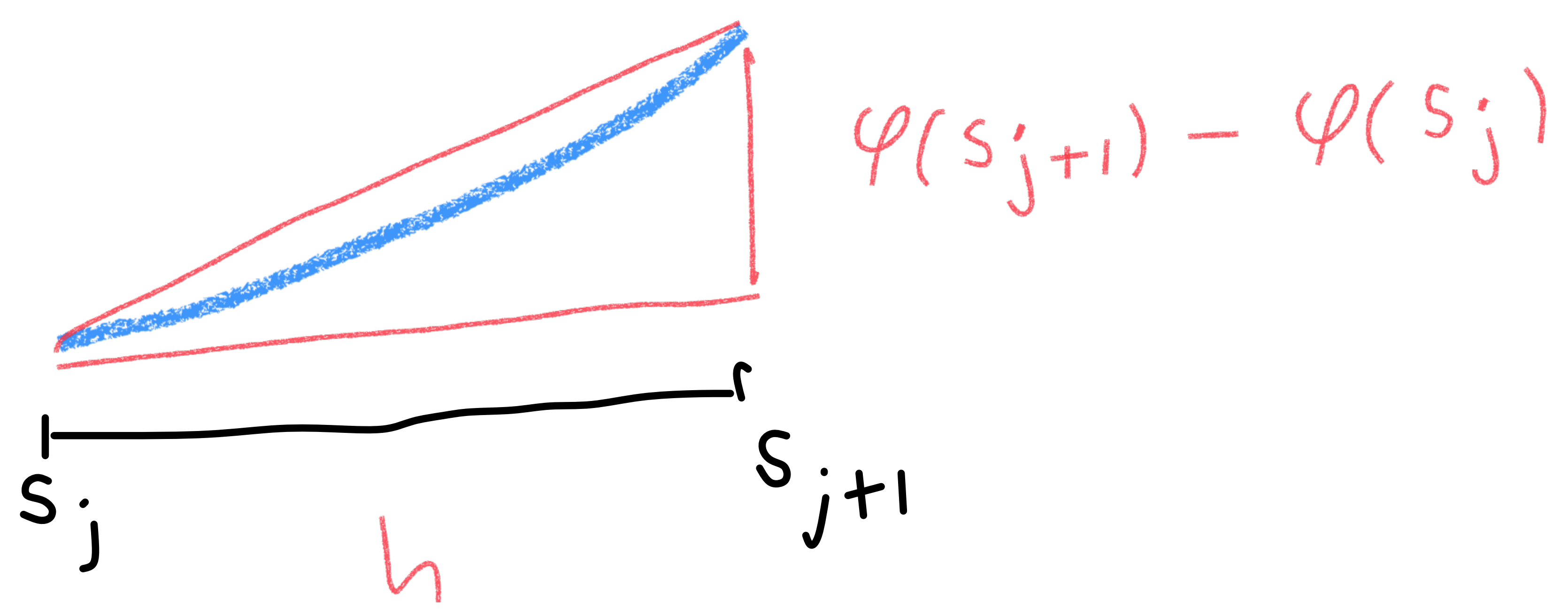
We give a theoretical motivation for the definition of this surface integral in special case.

$$\int_S f \, d\sigma \approx \sum_{j=0}^{N-1} f(s_j) \cdot \left[\text{length of graph arc from } f(s_j) \text{ to } f(s_{j+1}) \right]$$



The length of the arc from $f(s_j)$ to $f(s_{j+1})$ is circa

$$\sqrt{h^2 + \left(\varphi(s_{j+1}) - \varphi(s_j)\right)^2}$$



14

Graph surfaces: example in one variable

$$\sqrt{h^2 + \left(\varphi(s_{j+1}) - \varphi(s_j)\right)^2} = h\sqrt{1 + \left(\frac{\varphi(s_{j+1}) - \varphi(s_j)}{h}\right)^2}$$

$$\approx h\sqrt{1 + \partial_s \varphi(s_j)}$$

Hence $\displaystyle\int_S f \, d\sigma \approx \sum_{j=0}^{N-1} f(s_j) \cdot h\sqrt{1 + \partial \varphi(s_j)}$

$$\approx \int_a^b f(s)\sqrt{1 + \partial_s \varphi(s)} \, dx$$

Similar arguments justify the formula for general graph surfaces in two variables

15

General surfaces

16

General surfaces

Definition

Let $\Omega \subseteq \mathbb{R}^2$ be a domain and $\Phi : \Omega \to \mathbb{R}^3$ differentiable.
Consider the surface

$$S := \{ \Phi(s,t) \mid (s,t) \in \Omega \}$$

Then we call Φ a parameterization of the surface S

We say that Φ is regular if

$$\partial_s \Phi(s,t) \times \partial_t \Phi(s,t) \neq 0 \quad \text{for all } (s,t) \in \Omega$$

Compare to regular parameterizations of curves: $\dot{\gamma}(t) \neq 0$

17

General surfaces

Let $f : S \longrightarrow \mathbb{R}$ be a scalar field defined over the surface S,
The integral of f over S is :

$$\iint_S f \, d\sigma := \iint_\Omega f(\Phi(s,t)) \cdot \left\| \partial_s \Phi \times \partial_t \Phi \right\| \, ds \, dt$$

With general Φ we can describe a large range of surfaces :-)

But $\| \partial_s \Phi \times \partial_t \Phi \|$ has no particular form and can be anything

Finding a parameterization can be difficult

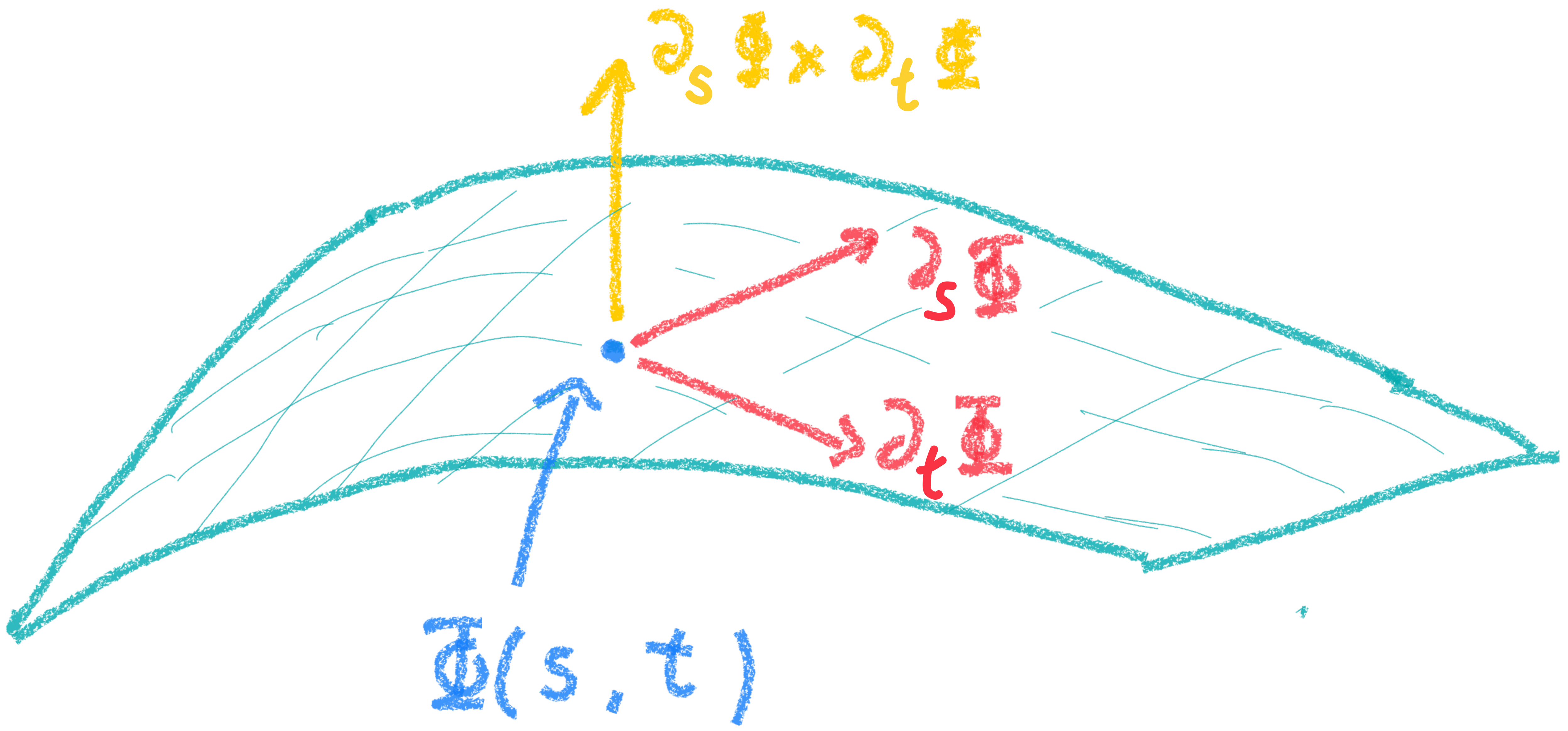
18

General surfaces

<u>Remark:</u>

$\partial_s \Phi(s,t)$ and $\partial_t \Phi(s,t)$ are tangential to S at $\Phi(s,t)$

So $\partial_s \Phi \times \partial_t \Phi$ is normal to S at $\Phi(s,t)$



19

General surfaces

One more remark:
different regular parameterizations may describe the same surface

$$\Phi : \Omega \longrightarrow \mathbb{R}^3, \qquad \Psi : \mathcal{O} \longrightarrow \mathbb{R}^3$$

with $\quad S = \Phi(\Omega) = \Psi(\mathcal{O}),\quad$ then

$$\iint_S f \, d\sigma = \iint_\Omega f(\Phi(s,t)) \cdot \| \partial_s \Phi \times \partial_t \Phi \| \, ds\, dt$$

$$= \iint_{\mathcal{O}} f(\Psi(s,t)) \cdot \| \partial_s \Psi \times \partial_t \Psi \| \, ds\, dt$$

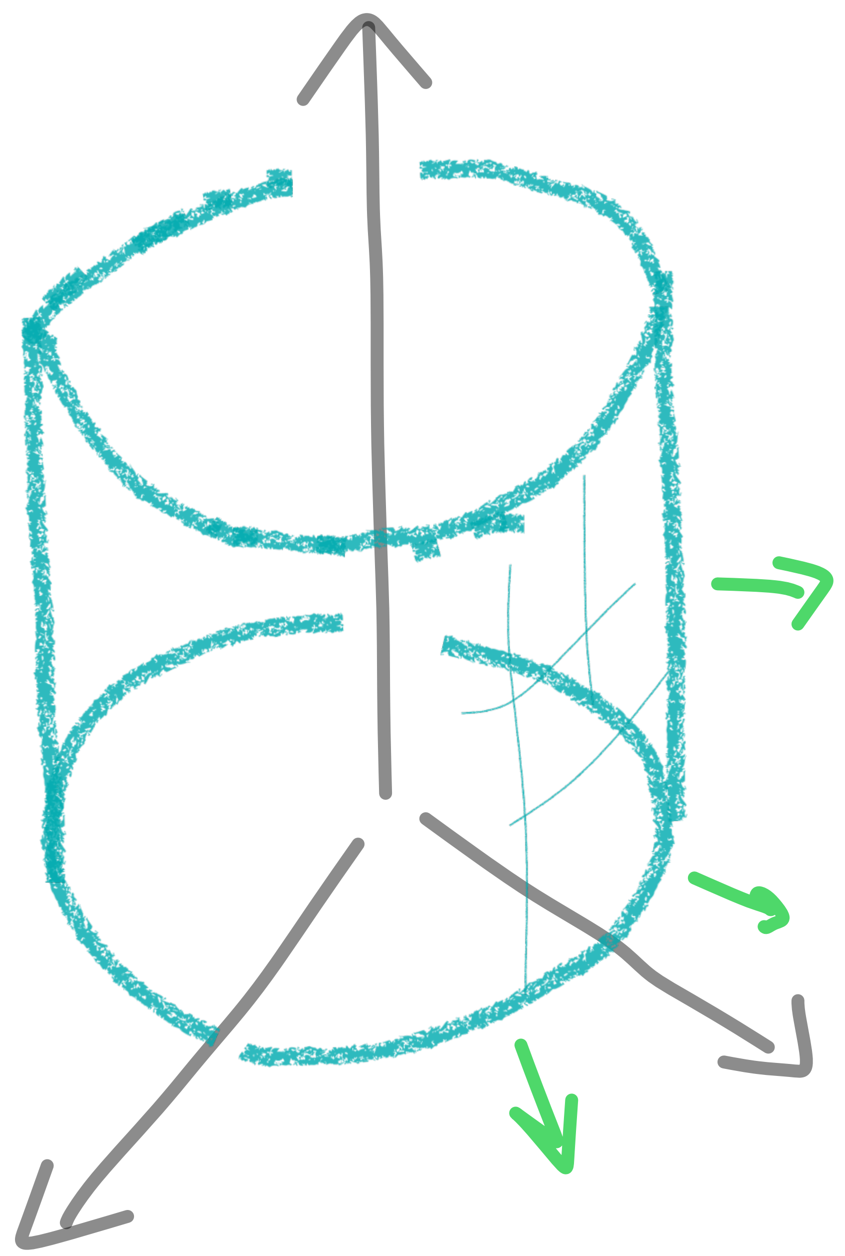
20

Examples of surfaces

21

Examples of surfaces

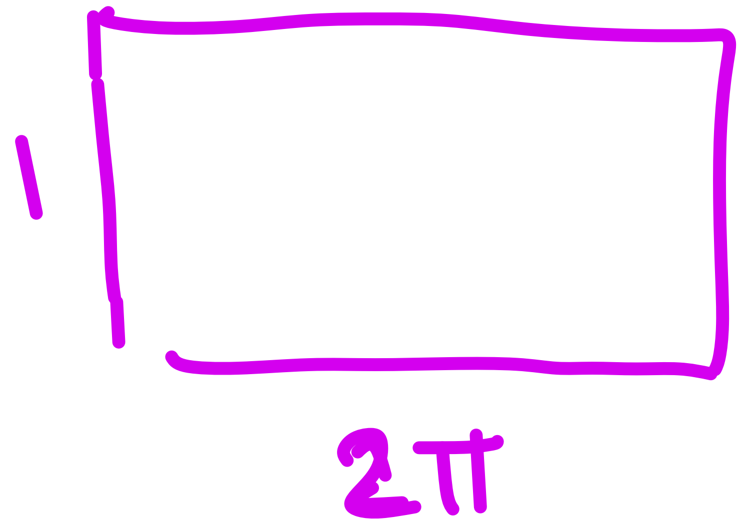
Cylinder shell



$$S = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 < x_3 < 1, \quad x_1^2 + x_2^2 = 1 \}$$

$$\Omega = \{ (t, z) \in \mathbb{R}^2 \mid 0 < z < 1, \quad 0 \leq t < 2\pi \}$$

$$\Phi : \Omega \longrightarrow \mathbb{R}^3, \qquad (t, z) \longmapsto (\cos t, \sin t, z)$$

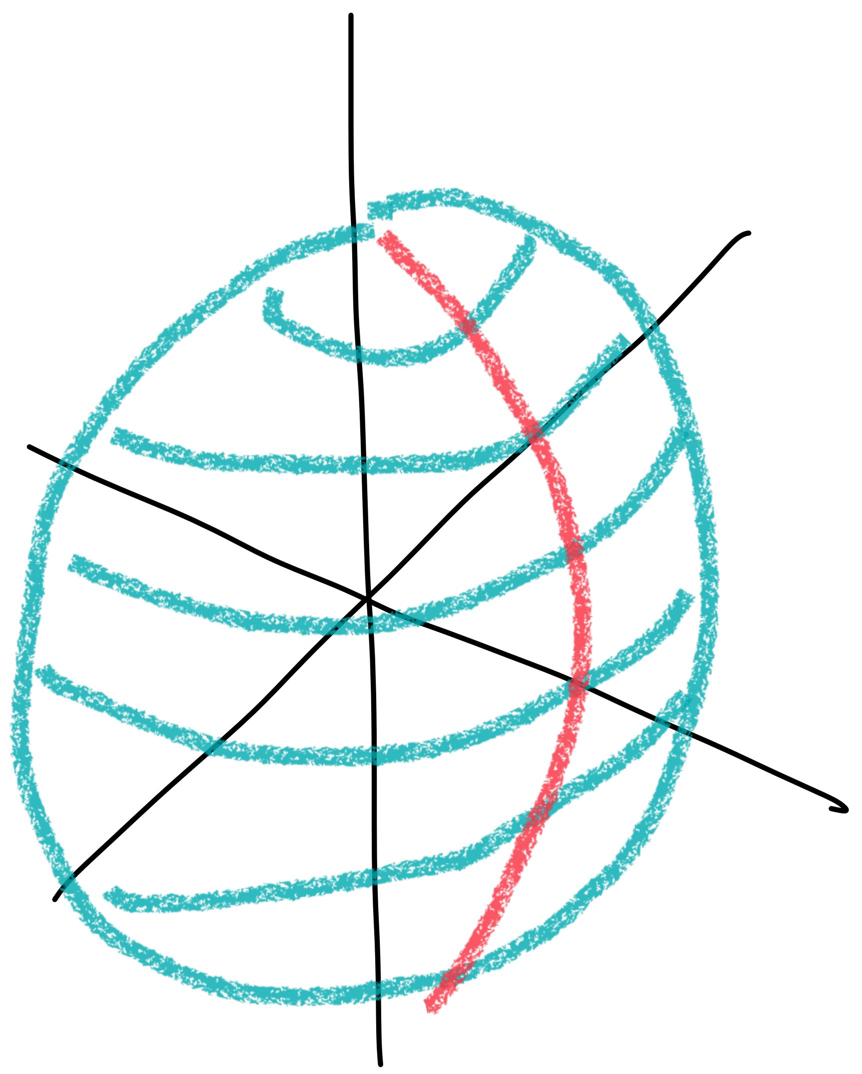


$$\partial_t \Phi \times \partial_z \Phi = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}$$

22

Examples of surfaces

Sphere



We describe the sphere in spherical coordinates

$$S = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \}$$

$$\Omega = [0, 2\pi] \times [0, \pi]$$

θ latitude $\quad$ φ longitude

$$\Phi : \Omega \to \mathbb{R}^3, \quad (\theta, \varphi) \mapsto (\cos\theta \sin\varphi, \ \sin\theta \sin\varphi, \ \cos\varphi)$$

23

Examples of surfaces

Sphere

$$\Phi : \Omega \to \mathbb{R}^3, \quad (\theta, \varphi) \mapsto \left(\cos\theta \sin\varphi , \ \sin\theta \sin\varphi , \ \cos\varphi \right)$$

$$\partial_\theta \Phi \times \partial_\varphi \Phi = \begin{pmatrix} -\sin\theta \sin\varphi \\ \cos\theta \sin\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\theta \cos\varphi \\ \sin\theta \cos\varphi \\ -\sin\varphi \end{pmatrix}$$

$$= \begin{pmatrix} -\cos\theta \sin^2\varphi \\ -\sin\theta \sin^2\varphi \\ -\sin^2\theta \left(\sin\varphi \cos\varphi \right) - \cos^2\theta \left(\sin\varphi \cos\varphi \right) \end{pmatrix} = - \begin{pmatrix} \cos\theta \sin^2\varphi \\ \sin\theta \sin^2\varphi \\ \sin\varphi \cos\varphi \end{pmatrix}$$

$$\left\| \partial_\theta \Phi \times \partial_\varphi \Phi \right\| = \sqrt{\cos^2\theta \sin^4\varphi + \sin^2\theta \sin^4\varphi + \sin^2\varphi \cos^2\varphi} = \sin\varphi$$

24

Examples of surfaces

Sphere

Surface area of unit sphere in 3D space

$$\int_S 1 \, d\sigma = \int_0^{2\pi} \int_0^{\pi} 1 \sin(\varphi) \, d\varphi \, d\theta$$

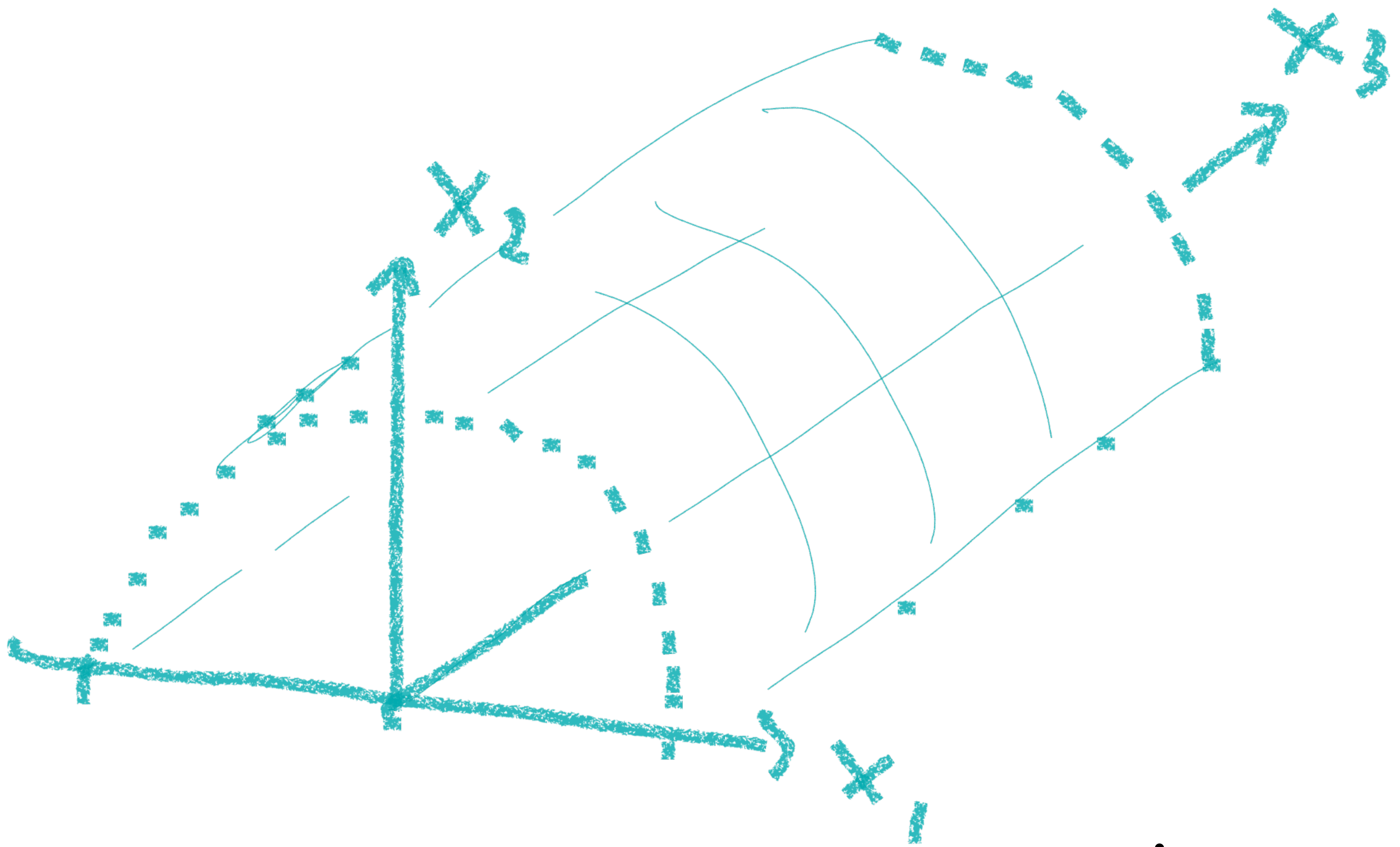
$$= \int_0^{2\pi} \Big[-\cos\varphi \Big]_0^{\pi} d\theta = 2\pi \cdot 2 = 4\pi$$

25

Examples of surfaces

Half-pipe

$$S := \{ \vec{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1, \quad x_2 > 0, \quad 0 < x_3 < 7 \}$$



We propose the following parameterization

$$\Omega = (-1,1) \times (0,7) \qquad \Phi(s,t) = \left(s, \sqrt{1-s^2}, t \right)$$

26

Examples of surfaces

Half-pipe

$$\partial_s \Phi(s,t) = \left(1, -s(1-s^2)^{-\frac{1}{2}}, 0\right) \qquad \partial_t \Phi(s,t) = (0, 0, 1)$$

$$\|\partial_s \Phi(s,t) \times \partial_t \Phi(s,t)\| = \left\|\left(-s(1-s^2)^{-1/2}, -1, 0\right)\right\| = \dots = \frac{1}{\sqrt{1-s^2}}$$

Find the integral over S of $f(x_1, x_2, x_3) = x_1^2 x_2 e^{x_3}$

$$\int_S f \, d\sigma = \int_{-1}^{1} \int_0^7 s^2 \sqrt{1-s^2}\, e^t \cdot \frac{1}{\sqrt{1-s^2}} \, dt\, ds$$

$$= \int_{-1}^{1} \int_0^7 e^t s^2 \, dt\, ds = \frac{2}{3}(e^7 - 1)$$

27

Integrals of vector fields over surfaces

28

Integrals of vector fields over surfaces

Let $\Phi : \Omega \to \mathbb{R}^3$ be a differentiable regular parameterization of a surface S. Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field

The surface integral of $\vec{F}$ along S is

$$\iint_S \vec{F}\, d\sigma := \iint_\Omega \vec{F} \cdot \left(\partial_s \Phi \times \partial_t \Phi \right) ds\, dt$$

produit mixte
triple product

29

Integrals of vector fields over surfaces

Expression as a surface integral of a scalar field:

$$= \iint_{\Omega} \vec{F}(\Phi(s,t)) \cdot \underbrace{\frac{\left(\partial_s \Phi(s,t) \times \partial_t \Phi(s,t)\right)}{\|\partial_s \Phi(s,t) \times \partial_t \Phi(s,t)\|}}_{\text{normalized normal vector}} \|\partial_s \Phi(s,t) \times \partial_t \Phi(s,t)\| \, ds \, dt$$

$$= \iint_{\Omega} F(x) \cdot \vec{n}(x) \, d\sigma$$

Outlook

31

Outlook

- We already knew how to integrate functions over domains in the plane
- We can now integrate functions over surfaces

$\Rightarrow$ Next: integral theorems of vector calculus over surfaces

32