EPFL - Fall 2024	Leonid Monin
Analysis III - 203 (a)	**Exercises**
Series 9	November 6, 2024

Note: several exercises are extracted from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Tip: To verify the Stokes theorem, proceed the following way:

1. *Sketch the surface* Σ*, then compute* $\operatorname{curl} F(x, y, z)$.
2. *Give a parametrization* $\sigma : \overline{A} \to \Sigma$ *of the surface* Σ *and give a normal vector. Add this vector to your sketch.*
3. *Express*
$$\iint_{\Sigma} \operatorname{curl} F \cdot ds$$
as a double integral where the bounds and the function to be integrated are explicitly indicated.
4. *Write* $\partial\Sigma$ *as the union of simple regular curves; for each of them, give a parametrization and indicate the direction of travel induced by the parametrization of* Σ *and the positive orientation of* ∂A.
5. *Express*
$$\int_{\partial\Sigma} F \cdot dl$$
as a sum of integrals where the bounds and the functions to be integrated are explicitly indicated.
6. *Verify the conclusion of the Stokes theorem for* Σ *and* F.

Exercise 1 (Ex 7.2 page 89)**.**
Verify Stokes' theorem for

$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^4,\ 0 \le z \le 1\}$ and $F(x, y, z) = (x^2 y, z, x)$.

Exercise 2 (Ex 7.5 page 89)**.**
Verify Stokes' theorem for

$\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4,\ x \ge 0,\ y \ge 0,\ 1 \le z \le \sqrt{3}\right\}$

and $F(x, y, z) = \big(0, z^2, 0\big)$.

Exercise 3 (Ex 7.7 page 89).
Verify the Stokes theorem for $F(x, y, z) = (0, x^2, 0)$ and Σ the triangle of vertices $(1, 0, 0), (2, 2, 0)$ and $(1, 1, 0)$.

Exercise 4 (Ex 7.6 page 89).
Verify the Stokes theorem for $F(x, y, z) = \big(0, 0, y + z^2\big)$ and

$$\Sigma = \Big\{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 4; x, y, z \geq 0; 0 \leq \arccos\frac{z}{2} \leq \arctan\frac{y}{x} \leq \frac{\pi}{2}\Big\}.$$

Note: exercises 5 and 6 are slightly anticipated in view of the lessons. You can wait the lesson of November, 25 to tackle them.

Exercise 5 (Ex 14.1 page 219).
Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function such that $f(x) = e^{(x-\pi)}$ over $[0, 2\pi[$.

1. Sketch the graph of f and the graph of f'.
2. Calculate the Fourier series Ff of the function f.
3. With the help of the Dirichlet theorem, compare Ff and f over $[0, 2\pi]$.
4. With the help of the two previous questions, show that

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{e^{\pi} - e^{-\pi}}.$$

Exercise 6 (Ex 14.2 page 220).
Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function such that $f(x) = (x - \pi)^2$ over $[0, 2\pi[$.

1. Sketch the graph of f and the graph of f'.
2. Calculate the Fourier series Ff of the function f.
3. With the help of the Dirichlet theorem, compare Ff and f over $[0, 2\pi]$.
4. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} \qquad \text{et} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$