EPFL - Fall 2024	Leonid Monin
Analysis III - 203 (a)	**Exercises**
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Note: several exercises are extracted from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Hint : In order to verify Green's theorem (e.g. exercises 1 and 2), apply the following steps:

1. Try to sketch the domain Ω and its edge $\partial\Omega$. Indicate the direction of the path $\partial\Omega$ so that the latter is positively oriented.
2. Compute $\text{curl}F(x,y)$.
3. Parametrize the domain Ω, and use this parametrization to compute
$$\iint_\Omega \text{curl}F\, dxdy.$$
4. Parametrize the edge $\partial\Omega$ of Ω, and use this parametrization to compute
$$\int_{\partial\Omega} F \cdot dl.$$
5. Verify the Green's theorem conclusion for Ω and F.

Exercise 1 (Ex 4.1 and Ex 4.2 page 41)**.**
Verify Green's theorem in the following cases:

1. $A = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}$ and $F(x,y) = (xy, y^2)$
2. $A = \left\{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\right\}$ and $F(x,y) = (x+y, y^2)$.

Exercise 2 (Ex 4.4i and Ex 4.5 page 42)**.**
Verify Green's theorem in the following cases:

1. $A = \left\{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 < 1\right\}$ and $F(x,y) = (-x^2 y, xy^2)$
2. $A = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1 \text{ et } x^2 - 4 < y < 2\right\}$ and $F(x,y) = (xy, y)$.

Exercise 3 (Ex 4.3 page 42).
Let $\Omega \subset \mathbb{R}^2$ be a triangle which has for vertices $(0,0), (0,1)$ and $(1,0)$. Let be $f(x,y) = y + e^x$. Compute:

1. $\displaystyle\int_{\Omega} \Delta f(x,y)\, dxdy$
2. $\displaystyle\int_{\partial\Omega} \left(\frac{\partial f}{\partial x}\nu_1 + \frac{\partial f}{\partial y}\nu_2 \right) dl$, where $\nu = (\nu_1, \nu_2)$ is the exterior unit vector normal to $\partial\Omega$.

Exercise 4 (Ex 4.9i page 43).
Let $\Omega \subset \mathbb{R}^2$ be a regular domain which has a positively oriented edge $\partial\Omega$. Let F, G_1 and G_2 be the vector fields defined as

$$F(x,y) = (-y, x), \quad G_1(x,y) = (0, x) \quad \text{and} \quad G_2(x,y) = (-y, 0).$$

Show that:

1. $\displaystyle \text{Area}(\Omega) = \frac{1}{2} \int_{\partial\Omega} F \cdot dl.$
2. $\displaystyle \text{Area}(\Omega) = \int_{\partial\Omega} G_1 \cdot dl.$
3. $\displaystyle \text{Area}(\Omega) = \int_{\partial\Omega} G_2 \cdot dl.$