Last time:

Divergence thm in $\mathbb{R}^3$

$$\boxed{\iiint_{\Omega} \operatorname{div} F(x,y,z)\, dx\,dy\,dz = \iint_{\partial\Omega} \langle F; \nu\rangle\, ds}$$

where ν is exterior unit normal vector field.

Version with parametrization of $\partial\Omega$

Let $\sigma: A \to \partial\Omega$ be a parametrization

then:

$$\iiint_{\Omega} \operatorname{div} (F(x,y,z)\, dx\,dy\,dz = \iint_{\partial\Omega} \langle F, \nu\rangle\, ds =$$

$$= \pm \iint_{A} \langle F(\sigma(u,v)); \sigma_u(u,v) \wedge \sigma_v(u,v)\rangle\, du\, dv$$

the sign depends on the direction of $\sigma_u \wedge \sigma_v$

if it is outward looking then $+$, if inward then $-$

Corollaries of divergence thm:

~~Area~~ Volume formulas, Green's identities in $\mathbb{R}^3$

Today: Stokes theorem. (Chapter 7)

Theorem 54 (~~Theorem~~ Stokes theorem)

Let Σ ~~be~~ $\subset \mathbb{R}^3$ be a piecewize-regular, orientable surface in $\mathbb{R}^3$ and let $F: \Sigma \to \mathbb{R}^3$ be a C^1-vector field then

$$\boxed{\iint_\Sigma \operatorname{rot} F \, ds = \int_{\partial\Sigma} F \cdot ds} \quad (*)$$

Remark ~~Both sides~~ The sign on both sides depends on the parametrization of Σ, and $\partial\Sigma$

We will spend some time defining compatible parametrizations.

Remark Statement analogous to fundamental thm of calculus.

How to choose compatible orientations?

<u>1st</u> Given ~~ori~~ parametrization $\sigma : A \to \Sigma$
we can produce parametrization of $\partial \Sigma$

For that we notice that the boundary $\partial\Sigma$ is given by ~~σ~~ $\sigma(\partial A)$ with removed

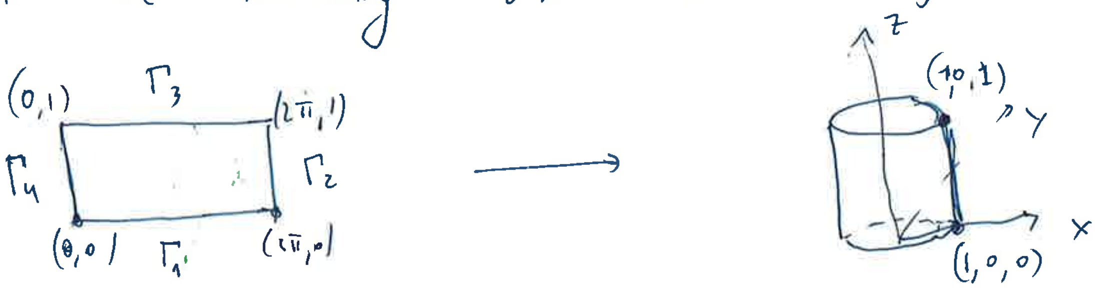
- Arcs which are traversed in two opposite directions
- parts which are reduced to a point.

<u>Two examples:</u> • Cylinder $\Sigma = \{(x,y,z) \mid x^2+y^2=1,\ z\in[0,1]\}$

We parametrize by $\sigma : \underbrace{[0,2\pi]\times[0,1]}_{= A} \longrightarrow \Sigma$

$$\sigma(\theta, z) = (\cos\theta, \sin\theta, z)$$

then the boundary ∂A has 4 regular components



We consider

$$\Gamma_1 = [(0,0), (2\pi, 0)]$$

$$\Gamma_2 = [(2\pi, 0), (2\pi, 1)]$$

$$\Gamma_3 = [(2\pi, 1), (0, 1)]$$

$$\Gamma_4 = [(0,1), (0,0)]$$

the boundary of Σ is given by $\sigma(\Gamma_1) \cup \sigma(\Gamma_3)$

and images of Γ_2 and Γ_4 cancel each other.

Example 2 Sphere: ~~$\{x \; x^2+y^2+z^2=1$~~ $\Sigma = \{(x,y,z) \mid x^2+y^2+z^2=1\}$

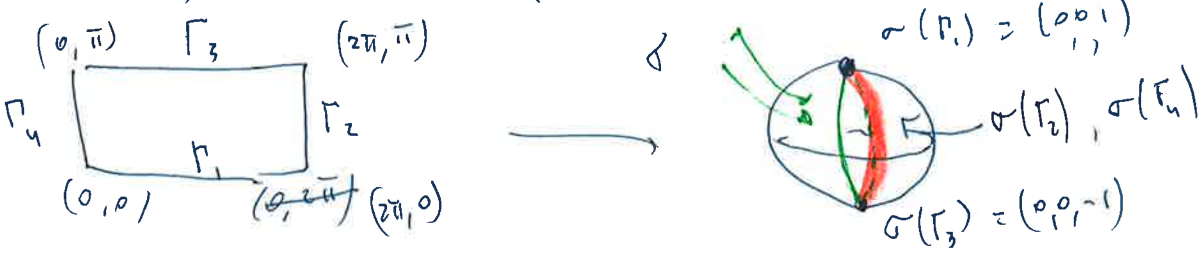
We parametrize Σ by $\sigma : [0,2\pi] \times [0,\pi] \to \Sigma$

$$(\theta, \varphi) \longmapsto (\cos\theta \sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$$

similarly, ∂A has 4 components:

$\Gamma_1 = [(0,0), (2\pi, 0)]$ $\quad \Gamma_2 = [(2\pi, 0), (2\pi, \pi)]$,

$\Gamma_3 = [(2\pi, \pi), (0, \pi)]$ $\quad \Gamma_4 = [(0,\pi), (0,0)]$



Note that

$$\sigma(\Gamma_1) = pt \quad , \quad \sigma(\Gamma_3) = pt$$

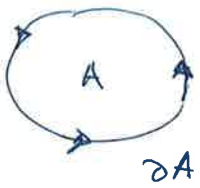
and $\sigma(\Gamma_2)$ and $\sigma(\Gamma_4)$ cancel each other

so $\partial \Sigma^2 = \emptyset$

Now we use the parametrization $\sigma : A \to \Sigma$ to get parametrization of $\partial \Sigma$.

For this choose a parametrization

$\gamma : [a, b] \to \partial A$ s.t. γ is positively oriented.



then we parametrize $\partial \Sigma$ via composition:

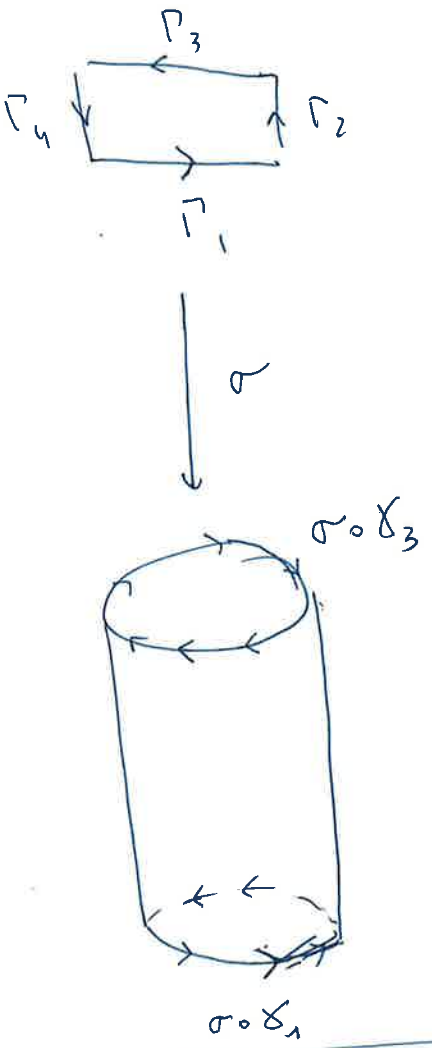
$$\underbrace{\sigma \circ \gamma} : [a, b] \longrightarrow \partial \Sigma$$

to make this parametrization regular we remove curves transversed ~~to~~ in oposite directions and points.

In the examples above:

- For cylinder: ~~$\Gamma_1 = \gamma_1(t)$~~

Γ_1 parametrized by ~~$\Gamma : [0, 2\pi]$~~ as $\gamma(t) = (\sin t, \cos t, 0)$ with $\forall\, t \in [0, 2\pi]$



Γ_2 parametrized by $\gamma(t) = (2\pi, t)$ with $t \in [0, 1]$

Γ_3 parametrized by $\gamma(t) = (2\pi - t, 1)$ $t \in [0, 2\pi]$

Γ_4 parametrized by $\gamma(t) = (0, 1, -t)$ $t \in [0, 1]$

To get right signs in $(*)$ • we chose parametrization $\sigma : A \to \Sigma$ (this gives the sign of $\iint_\Sigma \text{rot}\, F \cdot ds$)

- We chose positive orientation γ of ∂A
- We parametrize $\partial\Sigma$ via $\underline{\sigma \circ \gamma}$

(this gives the sign of $\int_{\partial\Sigma} F \cdot ds$)

Recap for vector calculus

- We discussed functions and vector fields and some differential operators with them:

$$\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right)$$

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

$$\operatorname{div} F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$$

$$\operatorname{rot} F = \begin{cases} \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} & \text{in } \mathbb{R}^2 \\[2ex] \left(\dfrac{\partial F_3}{\partial x_2} - \dfrac{\partial F_2}{\partial x_3}, \dfrac{\partial F_1}{\partial x_3} - \dfrac{\partial F_3}{\partial x_1}, \dfrac{\partial F_2}{\partial x_1} - \dfrac{\partial F_1}{\partial x_2}\right) & \text{in } \mathbb{R}^3 \end{cases}$$

- Then we talked about curve integrals on vector fields coming from potentials

Theorem 3.3 $\Omega \subset \mathbb{R}^n$ - regular and $F \in C(\Omega, \mathbb{R}^n)$

then the following are equivalent:

(i) F comes from potential on Ω

(ii) $\forall$ ~~simple~~ regular closed curve $\Gamma \subset \Omega$

we have $\int_\Gamma F \cdot dl = 0$

(iii) $\forall$ pair of regular curves ~~wit~~ between points A and B we get:

$$\int_{\Gamma_1} F \cdot dl = \int_{\Gamma_2} F \cdot dl$$

Then we talked about 3 Analogous thm.

Green's theorem (and divergence thm in $\mathbb{R}^2$)

Ω - regular domain in $\mathbb{R}^2$

- $\iint_\Omega \operatorname{rot} F \, dx\,dy = \int_{\partial\Omega} F \cdot dl$ ⟵ with positive parametr.

- $\iint_\Omega \operatorname{div} F \, dx\,dy = \int_{\partial\Omega} \langle F, \nu \rangle \, dl$ ⟵ outward unit normal

· Divergence thm in $\mathbb{R}^3$:

Theorem 6.2. Ω - regular in $\mathbb{R}^3$, $F \in C^1(\Omega, \mathbb{R}^3)$

then

$$\iiint_\Omega \operatorname{div} F \, dx\,dy\,dz = \iint_{\partial\Omega} \langle F \cdot \nu \rangle \, ds$$

outward normal.

· Stokes theorem

Theorem 7.1 $\Sigma \subset \mathbb{R}^3$ regular, orientable surface

$F: \Sigma \to \mathbb{R}^3$ is C^1 then

$$\iint_\Sigma \operatorname{rot} F \cdot ds = \int_{\gamma\Sigma} F \cdot dl$$

with compatible parametrizations