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Last Lectures:

Fourier series:

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right)$$

Theorem Dirichlet $f: \mathbb{R} \to \mathbb{R}$ – T-periodic piecewise continuous

then
$$Ff(x) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{f(x-\varepsilon) + f(x+\varepsilon)}{2} \quad \forall x \in \mathbb{R}$$

In particular If f is continuous

then $Ff(x) = f(x)$ for $\forall x \in \mathbb{R}$

Corollary • If f is even i.e. $f(x) = f(-x)$ then

$b_n = 0$ $\quad f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{2\pi n}{T} x\right)$.

• If f is odd i.e. $-f(x) = f(-x)$ then $a_n = 0$ and $Ff(x) = \sum b_n \sin\left(\frac{2\pi n}{T} x\right)$

Recall if function is defined on an interval we can extend it to a periodic function;

We can have a variation of this construction;

Let $f : [0, L] \to \mathbb{R}$ be a ~~con~~ piecewise continuous function then we can define:

- f_{even} - even piecewise cont. function periodic with period $2L$ s.t. $f(x) = f_{even}(x)$ $\forall x \in (0, L)$

- f_{odd} - odd $2L$-periodic function s.t. piecewise cont. $f(x) = f_{odd}(x)$ $\forall x \in (0, L)$

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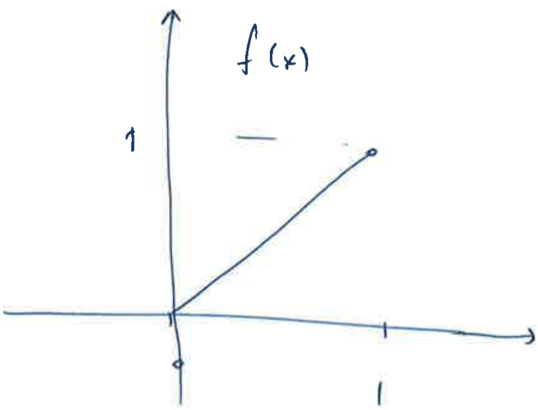
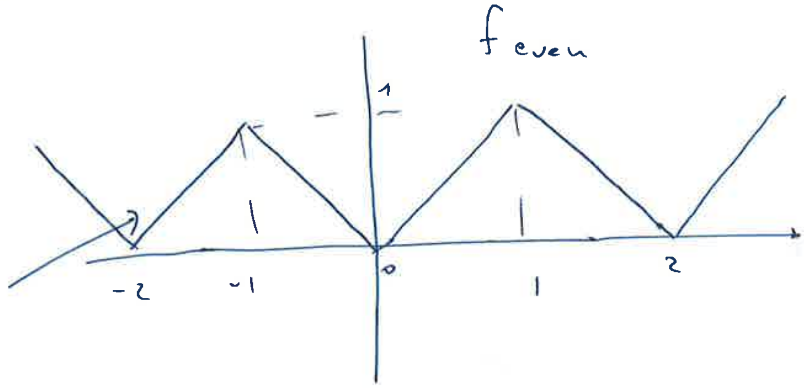
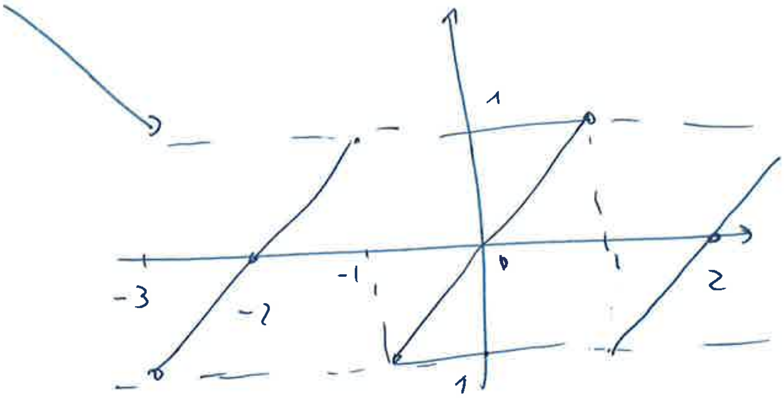
<u>Example</u>. $f(x) : (0,1) \to \mathbb{R}$

$f(x) = x$ then



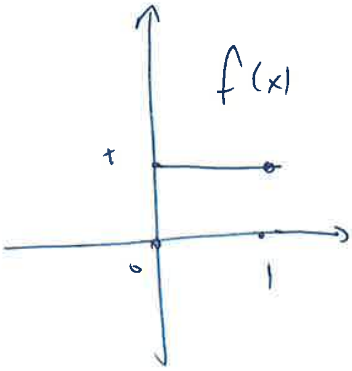


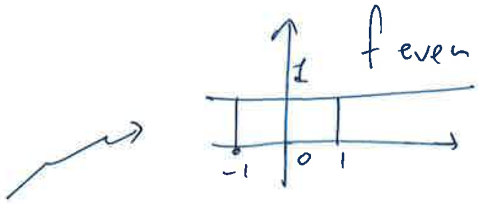


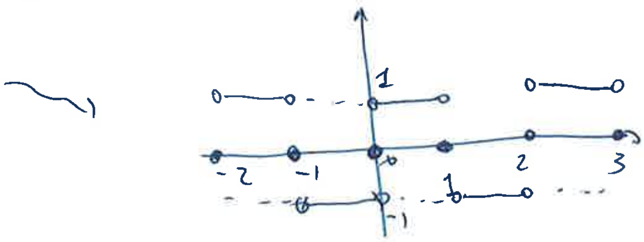
• $f(x) : (0,1) \to \mathbb{R}$

$f(x) = 1$ then $f_{even}(x) = 1 \quad \forall x \in \mathbb{R}$

$$f_{odd}(x) = \begin{cases} 1 & x \in (2k, 2k+1) \\ -1 & x \in (2k-1, 2k) \\ 0 & x \text{ is integer} \end{cases}$$







this leads to sin and cos Fourier series:

Corollary 14.8 Let $f: [0, L]$ be a piecewise continuous function then

we define cos - Fourier series as

$$\boxed{\begin{aligned} F_c f(x) &= \frac{a_0}{x} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n}{L} x\right); \\ \text{where } a_n &= \frac{2}{L} \int_0^L f(y) \cos\left(\frac{\pi n}{L} y\right) dy \end{aligned}}$$

Moreover, for $\forall x \in (0, L)$ we get:
if f is continuous

$$F_c f(x) = f(x).$$

Corollary 14.9 $f: (0, L) \to \mathbb{R}$ piecewise cont., then

$$\boxed{\begin{aligned} F_s f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L} x\right) \text{ with} \\ b_n &= \frac{2}{L} \int_0^L f(y) \sin\left(\frac{\pi n}{L} y\right) dy \end{aligned}}$$

Moreover, if f is continuous we have

$$\boxed{F_s f(x) = f(x) \quad \forall x \in (0, L)}$$

To prove these corollaries:

Compare $F_c f$ with $F f_{even}$ and

$F_s f$ with $F f_{odd}$

and use Dirichlet theorem.

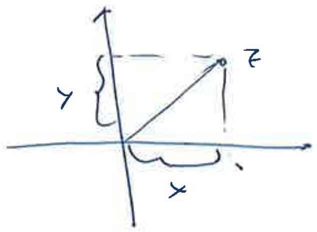
Chapter 15 Fourier transform

Quick recall of complex numbers:

A complex number $z \in \mathbb{C}$ ~~a+ib~~ can be written as

$$z = x + iy$$

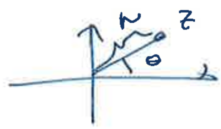
$x, y \in \mathbb{R}$, ~~tan~~ $i^2 = -1$



or

$$z = r \cdot e^{i\theta} = r\cos\theta + i r \sin\theta$$

Euler's formula.



$r \in \mathbb{R}_{\geq 0}$

$\theta \in \mathbb{R}$

So we get: for $x, y \in \mathbb{R}$

$$e^{x+iy} = e^x(\cos y + i \sin y)$$

Integrating complex functions:

$$\int_a^b f(x) + i \cdot g(x)\, dx = \int_a^b f(x)\, dx + i \int_a^b g(x)\, dx$$

Fourier transform.

<u>Definition</u> Let $f: \mathbb{R} \to \mathbb{R}$ be a function piecewise continuous and s.t.

$\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty$. then the Fourier transform of f is denoted by $\mathcal{F}(f)(\alpha)$ or $\hat{f}(\alpha)$ is defined as

$$\boxed{\mathcal{F}f(\alpha) = \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, e^{-i\alpha y}\, dy.}$$

Example

Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by:

$$f(x) = \begin{cases} \gamma & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

for some $\gamma \in \mathbb{R}$, $-\infty < a < b < \infty$

then

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}} \int_a^b \gamma \cdot e^{-i\alpha x}\, dx =$$

$$= \frac{\gamma}{\sqrt{2\pi}} \frac{1}{-i\alpha} e^{-i\alpha x} \Big]_{x=a}^{x=b} = \frac{\gamma}{\sqrt{2\pi}\, i\alpha} \left(e^{-i\alpha a} - e^{-i\alpha b} \right).$$

In general it is hard to compute Forier transform ~~explicit~~ directly often we use some properties of Fourier transform.

Remark. Sometimes the coefficient is taken to be 1 or $\frac{1}{2\pi}$ and $e^{-i\alpha y}$ is replaced with $e^{-2\pi i \alpha y}$.

Theorem 15.2 (Some properties of Fourier transform).

Let f, g be piecewise cont. functions s.t.

$$\int_{-\infty}^{+\infty} |f|\, dx < \infty, \quad \int_{-\infty}^{+\infty} |g(x)|\, dx < \infty$$ then we have

(i) $\hat{f} : \mathbb{R} \to \mathbb{C}$ is continuous and

$$\lim_{|\alpha| \to \infty} |\hat{f}(\alpha)| = 0$$

(ii) (Linearity) $\boxed{\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)}$

for any $a, b \in \mathbb{R}$

(iii) Let f be $C^1(\mathbb{R})$ and $\int_{-\infty}^{+\infty} |f'(x)|\, dx < \infty$

then $\boxed{\mathcal{F}(f')(\alpha) = i\alpha \cdot \mathcal{F}f(\alpha) \quad \forall \alpha}$

Moreover if $f \in C^n(\mathbb{R})$ and $\int_{-\infty}^{+\infty} |f^{(k)}(x)| dx < \infty$

for any $0 \le k \le n$ then

$$\boxed{\mathcal{F}(f^{(n)})(\alpha) = (i\alpha)^n \mathcal{F}(f)(\alpha) \quad \forall \alpha \in \mathbb{R}}$$

(iv) Assume that $\int_{-\infty}^{+\infty} |x^n f(x)| dx < \infty$ then

$$\boxed{\mathcal{F}(x^n f)(\alpha) = i^n \mathcal{F}^{(n)}(f)(\alpha) = i^n \cdot \frac{d^n}{d\alpha^n}[\hat{f}(\alpha)]}$$

for $n = 1$: $\underline{\mathcal{F}(x \cdot f)(\alpha) = i \, \mathcal{F}'(f)(\alpha) = i \frac{d}{d\alpha} \hat{f}(\alpha)}$

(vii) (Plancherel identity) If $\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$

then $$\boxed{\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |\hat{f}(\alpha)|^2 d\alpha}$$