Recall

$$\left\{ \sin kx \cdot \frac{2\pi}{T}, \quad \cos kx \cdot \frac{2\pi}{T} \right\} \quad \text{form}$$

„orthonormal basis" of space of T-periodic functions.

The inner product is given by $\langle f, g \rangle = \frac{2}{T} \int_0^T f \cdot g \, dx$

Def. Fourier series

$$n \geq 0 \qquad a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{2\pi}{T} nx\right) dx$$

$$n > 0 \qquad b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi}{T} nx\right) dx$$

$$F_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{2\pi}{T} nx\right) + b_n \sin\left(\frac{2\pi}{T} nx\right) \right]$$

Fourier series of order N.

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi}{T} nx\right) + b_n \sin\left(\frac{2\pi}{T} nx\right) \right].$$

Theorem (Dirichlet)

For $f : \mathbb{R} \to \mathbb{R}$ T-periodic, piecewise regular

then for every $x \in \mathbb{R}$

$Ff(x)$ converges and $Ff(x) = \underbrace{\lim_{t \to 0} \frac{f(x-t)+f(x+t)}{2}}$

If f is continuous at x, this lim is equal to $f(x)$.

Example Let $f(x) = \begin{cases} -1 & -1 \leq x < 0 \\ 1 & 0 \leq x < 1 \end{cases}$

extended by 2-periodicity.

Then
$$a_0 = \frac{2}{T}\int_0^T f(x)\,dx = \int_{-1}^{1} f(x)\,dx = 0$$

$$a_n = \int_{-1}^{1} f(x)\cos\left(\frac{2\pi}{2}\cdot n x\right)dx = \int_{-1}^{0} -\cos(\pi n x)\,dx + \int_0^1 \cos(\pi n x)\,dx =$$

$$= -\frac{\sin(\pi n x)}{\pi n}\Big]_{-1}^{0} + \frac{\sin(\pi n x)}{\pi n}\Big]_0^1 = 0$$

$$b_n = \int_{-1}^{1} f(x) \sin\left(\frac{2\pi}{2} nx\right) dx = \int_{-1}^{0} -\sin(\pi n x)\, dx + \int_{0}^{1} \sin(\pi n x)\, dx$$

$$= \frac{\cos(\pi n x)}{\pi n}\Bigg]_{-1}^{0} - \frac{\cos(\pi n x)}{\pi n}\Bigg]_{0}^{1} =$$

$$= \frac{2}{\pi n}\left(1 - \cos(\pi n)\right) = \frac{2}{\pi n}\left(1 - (-1)^n\right) =$$

$$= \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$\Rightarrow \quad Ff(x) = \sum_{k=1}^{\infty} \frac{4}{(2k+1)\pi} \cdot \sin\left((2k+1)\cdot \pi x\right)$$

Now notice that $f(x)$ is piecewise regular, so we get:

$$Ff(x) = f(x) \quad \text{for } x \notin \mathbb{Z}$$

$$Ff(\cancel{x}k) = 40 \quad \text{for every } k \in \mathbb{Z}$$

In particular we get that

$$1 = f\left(\tfrac{1}{2}\right) = Ff\left(\tfrac{1}{2}\right) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k+1} \sin\left((2k+1)\frac{\pi}{2}\right)$$

$$= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1}$$

Dirichlet can produce a lot of non-trivial identities in this way. Another example: $f(x) = x^2$ on $[-\frac{1}{2}, \frac{1}{2})$

then $Ff(x) = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2} \cos(2\pi n x)$

$f\left(\frac{1}{2}\right) = \frac{1}{4} = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Some corollaries of Dirichlet:

If f is piecewise regular, T-periodic then Ff is T-periodic and:

1) if f is even $(f(x) = f(-x))$ then $b_n = 0$

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\cos\left(\frac{2\pi}{T} n x\right)\right)$$

2) if f is odd $(f(-x) = -f(x))$ then $a_n = 0$

and $Ff(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{T} n x\right)$.

We can use complex numbers to write Fourier series:

$$\boxed{Ff(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i \frac{2\pi n}{T} x} \quad \text{with} \quad c_n = \frac{1}{T}\int_0^T f(x)\, e^{-i \frac{2\pi n}{T} x}\, dx}$$

Complex Fourier coefficients are related to real in the following way:

<u>Proposition</u> Let f be T-periodic, piecewise ~~ge~~ regular

Then:

1. $\forall n \geq 1$ we have $c_n = \dfrac{a_n - i\, b_n}{2}$ and

$\forall n \leq -1$ we have $c_n = \dfrac{a_n + i\, b_n}{2}$ and

$c_0 = \dfrac{a_0}{2}$

2. $\forall n \geq 1$ $\quad a_n = c_n + c_{-n}$; $\quad b_n = i(c_n - c_{-n})$

$a_0 = \cancel{c_0}\ 2c_0$

3. $F_N f(x) = \displaystyle\sum_{n=-N}^{N} c_n e^{i \frac{2\pi}{T} n x}$ $\qquad Ff(x) = \displaystyle\sum_{n=-\infty}^{+\infty} c_n e^{i \frac{2\pi}{T} n x}$.

Idea of proof: $e^{-i\frac{2\pi}{T}nx} =$

$$= \cos\left(-\frac{2\pi}{T}nx\right) + i\sin\left(-\frac{2\pi}{T}nx\right).$$

Properties of Fourier series

Theorems 14.5, 14.6

Differentiation term by term:

Let $f: \mathbb{R} \to \mathbb{R}$ be a T-periodic function, continuous s.t. f' is piecewise regular. ~~then~~: And let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2\pi n}{T}x\right) + b_n \sin\left(\frac{2\pi n}{T}x\right) \right\}$$

be its Forier series then we have:

$$\sum_{n=1}^{\infty} \frac{2\pi n}{T} \left\{ b_n \cos\left(\frac{2\pi n}{T}x\right) - a_n \sin\left(\frac{2\pi n}{T}x\right) \right\}$$

$$= \lim_{t \to x} \frac{f'(x-t) + f'(x+t)}{2}$$

Integration

Integration term by term:

Let $f: \mathbb{R} \to \mathbb{R}$ be a piecewise-regular, T-periodic function with ~~series~~ Fourier series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right) \right\}$$

Then:

$$\boxed{\int_{x_0}^{x} f(t)\,dt = \int_{x_0}^{x} \frac{a_0}{2}\,dt + \sum_{n=1}^{\infty} \int_{x_0}^{x} \left\{ a_n \cos\left(\frac{2\pi n}{T} t\right) + b_n \sin\left(\frac{2\pi n}{T} t\right) \right\} dt}$$

~~for~~ every $x, x_0 \in [0, T]$

Theorem 14.7 (Parseval Identity) For $f: \mathbb{R} \to \mathbb{R}$ a T-periodic ~~reg~~ piecewise regular function we get:

$$\frac{2}{T} \int_0^T f(x)^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = 2 \sum_{n=-\infty}^{+\infty} |c_n|^2$$

Real Fourier coefficients

Complex Fourier coeff.

Example:

$$f(x) = \begin{cases} 0 & x = -\pi \\ \frac{x}{2} & x \in (-\pi, \pi) \\ 0 & x = \pi \end{cases}$$

1st. $f(x)$ is odd $\Rightarrow$ $a_n = 0$

b_n can be computed to be $\dfrac{(-1)^{n+1}}{n}$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{2} \cdot \sin(nx)\, dx$$

So we get:

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x^2}{4}\, dx = \frac{\pi^2}{6}$$

So by Parseval we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$