Fourier Series (Chapter 14)

Motivation: Want to have good basis for spaces of functions.

Example for the space of analytic functions $f: \mathbb{R} \to \mathbb{R}$ we can use monomials: $\{x^n\}_{n \in \mathbb{Z}_{\geq 0}}$

This is known as Taylor expansion (at 0):

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$

Fourier series work for space of periodic functions $f: \mathbb{R} \to \mathbb{R}$.

That is functions f, s.t. $\exists T \in \mathbb{R}$ with

$$f(x+T) = f(x) \quad \forall x \in \mathbb{R}$$

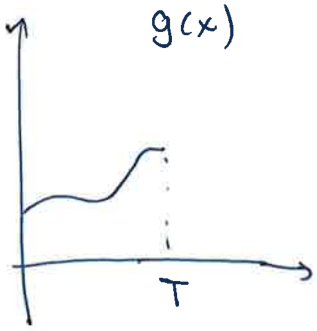
T is called the period of f.

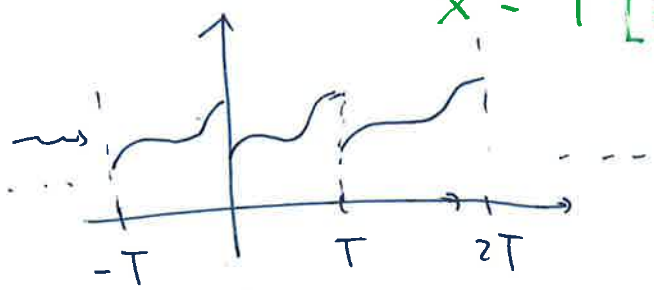
Example:

- $f(x) = \text{const}$ (periodic for every T)

- $g(x) : [0, T) \to \mathbb{R}$ some function

  define $f(x) = g\Big(\underbrace{T \cdot \big(\tfrac{x}{T} - \lfloor \tfrac{x}{T} \rfloor\big)}_{= \, x - T \lfloor \frac{x}{T} \rfloor}\Big)$





- $\cos\left(\frac{2\pi}{T} kx\right)$, $\sin\left(\frac{2\pi}{T} kx\right)$

- f, g – T-periodic, $\Rightarrow$

  $(f+g)$, $(f \cdot g)$, f/g are T-periodic

  if $g(x) \neq 0 \ \forall x$

In particular we get:

Proposition Let $V_T = \{ f : \mathbb{R} \to \mathbb{R},\ f \in C^\infty \text{ and } T\text{-periodic} \}$ then V_T is a vector space. Can define an inner product on V_T via:

$$\boxed{\langle f, g \rangle := \frac{2}{T} \int_0^T f(x) \cdot g(x) \, dx}$$

Note that if f, g are T-periodic then

$$\int_0^T f(x)\cdot g(x)\,dx = \int_{a}^{a+T} f(x)\,g(x)\,dx \quad \text{For any } a\in\mathbb{R}.$$

The main slogan of Fourier analysis:

$$\left\{ \sin\left(\frac{2\pi}{T}kx\right);\ \cos\left(\frac{2\pi}{T}mx\right)\right\}_{k,m\in\mathbb{Z}_{>0}}$$

Form an <u>orthonormal basis</u> of V_T

1. Orthogonality of $\sin\left(\frac{2\pi}{T}kx\right)$, $\cos\left(\frac{2\pi}{T}mx\right)$:

$$\left\langle \cos\left(\frac{2\pi}{T}kx\right), \cos\left(\frac{2\pi}{T}mx\right)\right\rangle = \frac{2}{T}\int_0^T \cos\left(\frac{2\pi}{T}kx\right)\cos\left(\frac{2\pi}{T}mx\right)dx$$

$$= \frac{2}{T}\int_0^T \frac{1}{2}\left(\cos\frac{2\pi}{T}(k-m)x + \cos\frac{2\pi}{T}(k+m)x\right)dx =$$

$$= \begin{cases} 1 & \text{if } k=m \\ 0 & \text{if } k\neq m \end{cases}$$

Similarly:

$$\left\langle \sin\left(\frac{2\pi}{T}\cdot kx\right), \sin\left(\frac{2\pi}{T} mx\right)\right\rangle = \begin{cases} 1 & k=m \\ 0 & \text{otherwise} \end{cases}$$

$$\left\langle \sin\left(\frac{2\pi}{T} kx\right), \cos\left(\frac{2\pi}{T} mx\right)\right\rangle = 0 \quad \forall\, m, k$$

~~We will see~~

So, $\left\{ \sin\left(\frac{2\pi}{T} kx\right), \cos\left(\frac{2\pi}{T} mx\right)\right\}$ form an orthonormal system of vectors. We will see that it is a basis later

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Definition of Fourier series:

Definition 14.2 Let $f: \mathbb{R} \to \mathbb{R}$ be a T-periodic function assume also that f is integrable on $[0, T]$ (i.e. $\int_0^T |f(x)|\, dx < \infty$) then for $n \in \mathbb{N}$

we define:

$$\boxed{\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(x)\cdot\cos\left(\frac{2\pi n}{T} x\right) dx = \left\langle f(x), \cos\left(\frac{2\pi n}{T} x\right)\right\rangle \quad \text{for } n = 0,1,2\ldots \\ b_n &= \frac{2}{T}\int_0^T f(x)\cdot\sin\left(\frac{2\pi n}{T} x\right) dx = \left\langle f(x), \sin\left(\frac{2\pi n}{T} x\right)\right\rangle \quad \text{for } n = 1,2,3\ldots \end{aligned}}$$

- We will ~~denote~~ define Fourier series of order N to be

$$\boxed{F_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right)\right)}$$

- We will define Fourier series to be:

$$\boxed{Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right)\right)}$$

"basis decomposition" for f

Theorem 14.3 (Dirichlet theorem)

Let f be a piecewise regular and T-periodic. Let a_n, b_n, $Ff(x)$ be as before. Then the limit $Ff(x)$ exists for any $x \in \mathbb{R}$ and:

$$\boxed{Ff(x) = \frac{f(x+0) + f(x-0)}{2}} \quad \forall x \in \mathbb{R}$$

$f(x+0) := \lim_{y \to x,\, y > x} f(y)$ $\qquad$ $f(x-0) = \lim_{\substack{y \to x \\ y < x}} f(y)$

Corollary If ~~$f \in C^\infty$ then:~~ continuous, periodic

then $Ff(x) = f(x)$ for any $x \in \mathbb{R}$

Corollary If f - ~~co~~ piecewise regular, T-periodic

then Ff is T-periodic and:

1) If f is even $\big(f(x) = f(-x)\big)$ then $b_n = 0$

and: $$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T} x\right)$$

2) If f is odd $\big(\text{i.e. } f(-x) = -f(x)\big)$ then $a_n = 0$

and $$Ff(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T} x\right)$$

~~3) We get complex~~