Last time:

Discussed Fourier transform:

~~As well as~~

Def: Let $f: \mathbb{R} \to \mathbb{R}$ piecewise continuous function s.t.h $\int_{-\infty}^{+\infty} |f| \, dx < \infty$

then $$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-i\alpha y} \, dy$$

Some properties:

1. Additive: $\mathcal{F}(f+g)(\alpha) = \mathcal{F}(f)(\alpha) + \mathcal{F}(g)(\alpha)$

2. $\mathcal{F}(f')(\alpha) = i\alpha \, \mathcal{F}(f)(\alpha)$ $\left(\text{If } \int_{-\infty}^{+\infty} |f'(x)| \, dx < \infty\right)$

   or $\mathcal{F}(f^{(n)})(\alpha) = (i\alpha)^n \, \mathcal{F}(f)(\alpha)$ $\left(\text{If } \int_{-\infty}^{+\infty} |f^{(k)}(x)| \, dx < \infty, \; k \le n\right)$

3. If $f * g(x) := \int_{-\infty}^{+\infty} f(x-t) \cdot g(t) \, dt$

   then $\mathcal{F}(f * g) = \sqrt{2\pi} \, \mathcal{F}(f)(\alpha) \cdot \mathcal{F}(g)(\alpha)$.

~~#~~ Today: ~~Th~~ Inverse Fourier transform

+ Applications to ODE

Fourier transform is invertible:

Theorem 15.3

(i) Let f be a piecewise continuous function

s.t. $\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty$ and $\int_{-\infty}^{+\infty} |\hat{f}(\alpha)|\, d\alpha < \infty$

then we get:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha)\, e^{i\alpha x}\, d\alpha .}$$

Applications of Fourier Analysis

Solving ODE

there are two ~~styles~~ situations:

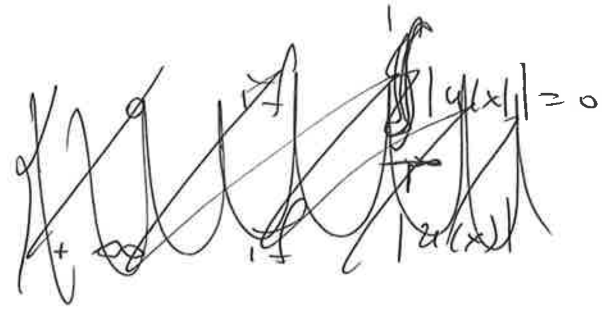
1. If we want to ~~sol~~ find a periodic solution to ODE we can use Fourier series

2. If we ~~as~~ want to find a solution $u: \mathbb{R} \to \mathbb{R}$ s.t. $\int_{-\infty}^{+\infty} |u(x)|\, dx < \infty$ then we can use Fourier transform.

Remark. Note that if $u: \mathbb{R} \to \mathbb{R}$ is periodic and then:

$$\int_{-\infty}^{+\infty} |u(x)|\, dx = \sum_{n=-\infty}^{+\infty} \int_{nT}^{nT+T} |u(x)|\, dx =$$

$$= \sum_{n=-\infty}^{+\infty} \int_0^T |u(x)|\, dx =$$



$$= \begin{cases} 0 & \text{if } \int_0^T |u(x)|\, dx = 0 \\ +\infty & \text{if } \int_0^T |u(x)|\, dx > 0 \end{cases}$$

If $u(x)$ is continuous, then:

$\int_0^T |u(x)|\, dx = 0$ implies that $u(x) = 0$

$\forall\, x \in [0, T]$ and hence $u(x) = 0 \quad \forall x \in \mathbb{R}$.

So for each particular problem

we can use only one method

Fourier series or Fourier transform.

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Example Let $f(x)$ be C^1-function which is 2π-periodic.

Find ~~solut~~ 2π-periodic ~~solutions~~ function s.t. $u'(x) - u(x) = f(x)$. $(*)$

~~Let~~ ~~$F(x)$~~ ~~to~~

Idea: Write $f(x)$ and $u(x)$, $u'(x)$ as Fourier series:

Let $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$

$$u(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos(nx) + B_n \sin(nx)\right)$$

Then $u'(x) = \sum_{n=1}^{\infty} \left(n B_n \cos(nx) - n \cdot A_n \sin(nx)\right)$

Plugging in $(*)$ we get:

$$-\frac{A_0}{2} + \sum_{n=1}^{\infty} (\underbrace{nB_n - A_n}_{\text{Unknown}}) \cos(nx) + (\underbrace{-nA_n - B_n}_{\text{Unknown}}) \sin(nx)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

We obtain a system of linear equations for A_n, B_n:

$$\boxed{A_0 = -a_0} \quad \text{and} \quad \begin{cases} nB_n - A_n = a_n \\ -nA_n - B_n = b_n \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \boxed{\begin{cases} A_n = \dfrac{-a_n - nb_n}{1+n^2} \\ B_n = \dfrac{-b_n + na_n}{1+n^2} \end{cases}}$$

This determines function u

$$\boxed{u(x) = -\frac{a_0}{2} + \sum \frac{-a_n - nb_n}{1+n^2} \cos(nx) + \frac{na_n - b_n}{1+n^2} \sin(nx)}$$

For instance if $f(x) = 3 + \cos(x) + 5\sin(2x) - \cos(2x)$

we get $a_0 = 3$, $a_1 = 1$, $\cancel{b}\,a_2 = -1$, $b_2 = 5$

$$\Rightarrow \quad A_0 = -3 \qquad A_1 = \frac{-a_1 \cancel{+} - b_1}{1+1^2} = \frac{1-0}{1+1^2} = 1/2$$

$$B_1 = \frac{-b_1 + 1\cdot a_1}{1+1^2} = \frac{0+1}{2} = \frac{1}{2}$$

$$A_2 = \frac{-a_2 - 2b_2}{1+2^2} = \frac{\cancel{1}\;1 - 10}{5} = -\frac{9}{5}$$

$$B_2 = \frac{-b_2 + 2a_2}{1+2^2} = \frac{-5 + 2\cdot(-1)}{5} = -\frac{7}{5}$$

So we get

$$\boxed{u(x) = -3 + \frac{1}{2}(\cos x) + \frac{1}{2}(\sin x) - \frac{9}{5}\cos(x) - \frac{7}{5}\sin(x).}$$

<u>Example</u> Find $u : \mathbb{R} \to \mathbb{R}$ s.t.

$$u''(x) + 2u(x) = x^2 e^{-x} \qquad (*)$$

We use Fourier transform as <u>$x^2 e^{-x}$ is not periodic.</u>

$$\mathcal{F}(u'' + 2u) = \mathcal{F}(u'') + 2\mathcal{F}(u) =$$

$$= (i\alpha)^2 \hat{u}(\alpha) + 2\hat{u}(\alpha) = (2 - \alpha^2)\hat{u}(\alpha)$$

$$\mathcal{F}(x^2 e^{-x}) = \mathcal{F}(x \cdot \underline{x e^{-x^2}}) = i \cdot \frac{d}{d\alpha} \mathcal{F}(x \cdot e^{-x^2})$$

Function from table of Fourier transforms

Table 15.5 (g)

$$= i \frac{d}{d\alpha} \left(\frac{-i\alpha}{2\sqrt{2}} \cdot e^{-\frac{\alpha^2}{4}} \right) = i \left(\frac{-i \cancel{\alpha}}{2\sqrt{2}} \cdot e^{-\frac{\alpha^2}{4}} + \right.$$

$$\left. + \frac{i\alpha}{\sqrt{2} \cdot 2} \cdot \left(-\frac{\alpha}{2}\right) \cdot e^{-\frac{\alpha^2}{4}} \right)$$

$$= \frac{1}{2\sqrt{2}} e^{-\frac{\alpha^2}{4}} - \frac{\alpha^2}{4\sqrt{2}} e^{-\frac{\alpha^2}{4}}$$

So we get:

$$(2-\alpha^2)\,\hat{u} = \frac{1}{\sqrt{2}\cdot 2}\cdot e^{-\frac{\alpha^2}{4}}\left(1-\frac{\alpha^2}{2}\right)$$

$$\Rightarrow \quad \hat{u} = \frac{e^{-\frac{\alpha^2}{4}}}{\sqrt{2}\cdot 2}\cdot\frac{1-\frac{\alpha^2}{2}}{2-\alpha^2} = \frac{e^{-\frac{\alpha^2}{4}}}{4\sqrt{2}} =$$

$$= \frac{1}{4}\mathcal{F}\left(e^{-x^2}\right) \quad \Rightarrow \quad \boxed{u(x) = \frac{1}{4}e^{-x^2}}$$

gives a solution of (*)

Table 15.5 (8)

Example. Find $u(x)$ s.t.

$$9u(x) + \int_{-\infty}^{+\infty} 8u(t)\cdot e^{-|x-t|}\,dt = e^{-|x|}$$

Note that $\displaystyle\int_{-\infty}^{+\infty} 8u(t)\cdot e^{-|x-t|}\,dt = u * e^{-|x|}$

convolution

Let's use Fourier transform:

~~we se~~

$$\mathcal{F}\left(9u + 8u * e^{-|x|}\right) = \mathcal{F}\left(e^{-|x|}\right)$$

$\Updownarrow$

$$9\hat{u} + 8\hat{u} \cdot \underbrace{\sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+\alpha^2}}_{\mathcal{F}\left(e^{-|x|}\right)} \cdot \sqrt{2\pi} = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+\alpha^2}$$

by table 15.5 (7).

$\Updownarrow$

$$\hat{u} \cdot \left(9 + 16 \cdot \frac{1}{1+\alpha^2}\right) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+\alpha^2}$$

$\Updownarrow$

$$\hat{u} \cdot \left(9 + 9\alpha^2 + 16\right) = \sqrt{\frac{2}{\pi}}$$

$$\Rightarrow \hat{u}(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{25 + 9\alpha^2} =$$

$$= \frac{1}{9}\sqrt{\frac{2}{\pi}} \cdot \frac{1}{\left(\frac{5}{3}\right)^2 + \alpha^2} = \frac{1}{9} \cdot \mathcal{F}\left[\frac{e^{-\frac{5}{3}|x|}}{\frac{5}{3}}\right] \Rightarrow$$

$$\Rightarrow u(x) = \frac{1}{9} \cdot \frac{3}{5} e^{-\frac{5}{3}|x|} = \boxed{\frac{1}{15} \cdot e^{-\frac{5}{3}|x|}}$$