Last time:

Fourier transform

Definition Let $f: \mathbb{R} \to \mathbb{R}$ be a function s.t. it is piecewise cont and $\int_{-\infty}^{+\infty} |f| \, dx < \infty$

then its Fourier transform is:

$$\mathcal{F}f(\alpha) = \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) \, e^{-i\alpha y} \, dy.$$

Remark It is a continuous version of Fourier series

(for that think of function f s.t.

$f(x) = 0 \quad \forall x \notin [a, b]$ and the corresponding periodic function).

Properties of Fourier transform:

Thm (15.2)

(i) $\hat{f}: \mathbb{R} \to \mathbb{C}$ is continuous and

$$\lim_{|\alpha| \to \infty} |\hat{f}(\alpha)| = 0$$

(ii) Linearity $\forall a, b \in \mathbb{R}$

$$\boxed{\mathcal{F}(af + bg)(\alpha) = a \cdot \mathcal{F}f(\alpha) + b \cdot \mathcal{F}g(\alpha)}$$

(iii) Fourier transform of derivatives:

$$\boxed{\begin{aligned} \mathcal{F}(f')(\alpha) &= i\alpha \cdot \mathcal{F}f(\alpha) \\ \mathcal{F}(f^{(n)})(\alpha) &= (i\alpha)^n \cdot \mathcal{F}f(\alpha). \end{aligned}}$$

(iv) Fourier transform of $x^n \cdot f$

Assume that $\int_{-\infty}^{+\infty} |x^n \cdot f| \, dx < \infty$ then:

$$\boxed{\mathcal{F}(x^n f)(\alpha) = i^n \cdot \mathcal{F}^{(n)}(f)(\alpha) = i^n \frac{d^n}{d\alpha^n} [\hat{f}(\alpha)]}$$

Example (fixed point of Fourier transform)

Q: Do we have function f with $\mathcal{F}f = f$

Consider functions $f(x) = e^{-\frac{x^2}{2}}$; $h(x) = x \cdot e^{-\frac{x^2}{2}}$
$= x f(x)$

Note: $\hat{h}(\alpha) = \mathcal{F}(xf) = i \cdot \hat{f}'(\alpha)$ ~~by condi~~ by (iv)

Also $f'(x) = -x \cdot e^{-\frac{x^2}{2}} = -h(x)$

We get:

$$\underbrace{\hat{f}'(\alpha) = -i \hat{h}(\alpha)}_{\text{by (iv)}} \underbrace{=}_{\text{by (ii)}} i \, \mathcal{F}(-h)(\alpha) = i \, \mathcal{F}(f')(\alpha) =$$

$$\underbrace{=}_{\text{by (iii)}} i \cdot (i\alpha \cdot \hat{f}(\alpha)) = -\alpha \hat{f}(\alpha)$$

So we get $\hat{f}'(\alpha) = -\alpha \cdot \hat{f}(\alpha)$ (*)

We can check directly that the solution (

to (*) is $c\, e^{-\frac{\alpha^2}{2}}$ for some constant c.

But $\hat{f}(0) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} f(x) \cdot e^{-i \cdot 0 \cdot x}\, dx =$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx = 1 \Rightarrow$$

$$\Rightarrow \quad c \cdot e^{0} = 1 \quad \Rightarrow \quad c = 1$$

We get $\boxed{\mathcal{F}\left(e^{-\frac{x^2}{2}}\right) = e^{-\frac{\alpha^2}{2}}}$.

<u>Plancherel identity:</u>

Thm (15.2)

(vii) If $\displaystyle\int_{-\infty}^{+\infty} |\hat{f}(x)|^2\, dx < \infty$ we get

$$\boxed{\int_{-\infty}^{+\infty} |f(x)|^2\, dx = \int_{-\infty}^{+\infty} |f(\alpha)|^2\, d\alpha}$$

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Example

Let $$f(x) = \begin{cases} 1 & -a \le 1 \le a \\ 0 & \text{otherwise} \end{cases}$$

then $$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \cdot e^{-i\alpha x}\, dx =$$

$$= \int_{-a}^{a} e^{-i\alpha x}\, dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin a\alpha}{\alpha}$$

Using Plancherel's identity:

$$\int_{-\infty}^{+\infty} |f(x)|^2\, dx = \int_{-\infty}^{+\infty} |\hat{f}(\alpha)|^2\, d\alpha \Rightarrow$$

$$\Rightarrow \int_{-a}^{a} 1\, dx = \int_{-\infty}^{+\infty} \left(\sqrt{\frac{2}{\pi}} \cdot \frac{\sin(a\alpha)}{\alpha} \right)^2 d\alpha$$

$$\Rightarrow 2a = \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{\sin^2(a\alpha)}{\alpha^2}\, d\alpha$$

Finally we get:

$$\boxed{\int_{-\infty}^{+\infty} \frac{\sin^2(a\alpha)}{\alpha^2}\, d\alpha = a\cdot\pi}$$ for $a=1$:

$$\boxed{\int_{-\infty}^{+\infty} \frac{\sin^2(\alpha)}{\alpha^2}\, d\alpha = \pi}$$

Something we seen & in the above example:

If function $f(x)$ is even or odd

We can complete its Fourier transform

easier.

Theorem 15.3

(ii) If f is even then

$$\mathcal{F}f(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(y)\cdot\cos(\alpha y)\,dy$$

(iii) If f is odd then:

$$\mathcal{F}f(\alpha) = -i\sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(y)\cdot\sin(\alpha y)\,dy.$$

Convolution and transform of convolution.

<u>Definition</u> For $f, g : \mathbb{R} \to \mathbb{R}$ s.t.

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty \; ; \quad \int_{-\infty}^{+\infty} |g(x)|\, dx < \infty$$

we define their convolution to be:

$$f * g(x) = \int_{-\infty}^{+\infty} f(x-t) \cdot g(t)\, dt = \int_{-\infty}^{+\infty} f(t)\, g(x-t)\, dt$$

Theorem (15.2)

(vi) Fourier of convolution:

For f, g as before we get:

$$\boxed{\mathcal{F}(f * g)(\alpha) = \sqrt{2\pi}\, \mathcal{F}(f) \cdot \mathcal{F}(g)\, (\alpha)}$$