

Note: the following correction (in french) has been prepared by Dr. David Strütt.

Exercise 5.

Function f is piecewise-defined and we can verify that:

$$\int_{-\infty}^{+\infty} |f(x)| dx = \int_{-1}^1 1 dx = 2 < +\infty.$$

Thus, the Fourier transform is well-defined. We compute:

$$\begin{aligned}\hat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx \\ &= \int_{-1}^1 e^{-i\alpha x} dx \\ &= -\frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\alpha x}}{i\alpha} \right]_{-1}^1 \\ &= \frac{1}{\sqrt{2\pi}} \frac{e^{i\alpha} - e^{-i\alpha}}{i\alpha} \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin \alpha}{\alpha}.\end{aligned}$$

Exercise 6.

- a) La fonction f est continue par morceaux et intégrable sur $\mathbb{R}$. Sa transformée de Fourier vaut

$$\begin{aligned}\hat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 \sqrt{\frac{\pi}{2}} e^{-i\alpha x} dx \\ &= \frac{1}{2} \left[\frac{1}{-i\alpha} e^{-i\alpha x} \right]_{-1}^1 = \frac{1}{2i\alpha} (e^{i\alpha} - e^{-i\alpha}) = \frac{\sin(\alpha)}{\alpha}.\end{aligned}$$

b) La fonction g est continue et intégrable sur $\mathbb{R}$. Sa transformée de Fourier vaut

$$\begin{aligned}
\hat{g}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x) e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-2}^0 \left(\pi + \frac{\pi}{2}x\right) e^{-i\alpha x} dx + \frac{1}{\sqrt{2\pi}} \int_0^2 \left(\pi - \frac{\pi}{2}x\right) e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-i\alpha} e^{-i\alpha x} \left(\pi + \frac{\pi}{2}x\right) \right]_{-2}^0 - \frac{1}{\sqrt{2\pi}} \int_{-2}^0 \frac{\pi}{-2i\alpha} e^{-i\alpha x} dx \\
&\quad + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-i\alpha} e^{-i\alpha x} \left(\pi - \frac{\pi}{2}x\right) \right]_0^2 - \frac{1}{\sqrt{2\pi}} \int_0^2 \frac{\pi}{2i\alpha} e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{\pi}{2} \left(\left[\frac{e^{-i\alpha x}}{-\alpha^2} \right]_0^2 - \left[\frac{e^{-i\alpha x}}{-\alpha^2} \right]_{-2}^0 \right) \\
&= \frac{\sqrt{\pi}}{2\sqrt{2}} \left(\frac{e^{2i\alpha} + e^{-2i\alpha}}{2} \frac{2}{-\alpha^2} - \frac{2}{-\alpha^2} \right) \\
&= \frac{\sqrt{\pi}}{-\alpha^2 \sqrt{2}} (\cos(2\alpha) - 1) = \sqrt{2\pi} \frac{\sin^2(\alpha)}{\alpha^2}.
\end{aligned}$$

Nous pouvons observer que $\hat{g}(\alpha) = \sqrt{2\pi} \hat{f}(\alpha)^2$. Ainsi $g = f * f$.