EPFL - Fall 2024	Veronica Montanaro
Analysis III - 203 (a)	**Correction**
Series 5	October 9, 2024

Note : For exercise 1 and 2, the correction will follow the steps described in the hint of Series 5.

Exercise 1 (Ex 4.1 and Ex 4.2 page 41)**.**

1. **Step 1.** We find
$$\operatorname{rot} F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -x.$$
We now switch to polar coordinates $x = r\cos\theta,\ y = r\sin\theta$. For polar coordinates the Jacobian of the transformation is equal to r. Then
$$\iint_\Omega \operatorname{rot} F \ dxdy = \int_0^1 \int_0^{2\pi} (-r\cos\theta)\, r \ drd\theta = 0.$$
Step 2. Let's evaluate $\int_{\partial\Omega} F \cdot dl$. Let $\partial\Omega$ be parametrized for $\gamma(\theta) = (\cos\theta, \sin\theta)$,
$$\begin{aligned}\int_{\partial\Omega} F \cdot dl &= \int_0^{2\pi} \left(\cos\theta\sin\theta, \sin^2\theta\right) \cdot (-\sin\theta, \cos\theta)\, d\theta \\ &= \int_0^{2\pi} \left(-\cos\theta\sin^2\theta + \cos\theta\sin^2\theta\right) d\theta = 0.\end{aligned}$$

2. **Step 1.** We find
$$\operatorname{rot} F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -1..$$
Switching to polar coordinates we end up with
$$\iint_\Omega \operatorname{rot} F \ dxdy = \int_0^1 \int_0^{2\pi} (-1)\, r \ drd\theta = -3\pi.$$
Step 2. Let's evaluate $\int_{\partial\Omega} F \cdot dl$. We split $\partial\Omega$ in two curves and have
$$\int_{\partial\Omega} F \cdot dl = \int_{\Gamma_0} F \cdot dl - \int_{\Gamma_1} F \cdot dl$$
with
$$\begin{gathered}\Gamma_0 = \{\gamma_0(\theta) = (2\cos\theta, 2\sin\theta) : \theta \in [0, 2\pi]\} \\ \Gamma_1 = \{\gamma_1(\theta) = (\cos\theta, \sin\theta) : \theta \in [0, 2\pi]\}.\end{gathered}$$
One finds that
$$\int_{\Gamma_0} F \cdot dl = \int_0^{2\pi} \left(2\cos\theta + 2\sin\theta, 4\sin^2\theta\right) \cdot (-2\sin\theta, 2\cos\theta)\, d\theta = -4\pi$$

$$\int_{\Gamma_1} F \cdot dl = \int_0^{2\pi} \big(\cos\theta + \sin\theta, \sin^2\theta\big) \cdot (-\sin\theta, \cos\theta)\, d\theta = -\pi.$$

As a consequence we obtain

$$\int_{\partial\Omega} F \cdot dl = \int_{\Gamma_0} F \cdot dl - \int_{\Gamma_1} F \cdot dl = -3\pi.$$

Exercise 2 (Ex 4.4i and Ex 4.5 page 42)**.**

1. **Step 1.** For the calculation of $\iint_A \operatorname{rot} F dxdy$ we find

   $$\operatorname{rot} F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = x^2 + y^2.$$

   We redefine A in polar coordinates

   $$A = \{(x,y) \in \mathbb{R}^2 : x = r\cos\theta, y = 1 + r\sin\theta, \theta \in [0, 2\pi], r \in (0,1)\}.$$

   This leads to

   $$\iint_A \operatorname{rot} F \; dxdy = \int_0^1 \int_0^{2\pi} \big[\, (r\cos\theta)^2 + (r\sin\theta + 1)^2 \,\big] r \; drd\theta = \frac{3\pi}{2}.$$

   Step 2. For the calculation of $\int_{\partial A} F \cdot dl$ we find ∂A is parametrized for

   $$\partial A = \{(x,y)\, \mathbb{R}^2 : x = \cos\theta, y = 1 + \sin\theta, \theta \in [0, 2\pi]\}$$

   so

   $$\int_{\partial A} F \cdot dl = \int_0^{2\pi} \big(- (1+\sin\theta)\cos^2\theta, (1+\sin\theta)^2 \cos\theta \big) \cdot (-\sin\theta, \cos\theta)\, d\theta = \frac{3\pi}{2}.$$

2. **Step 1.** We find $\operatorname{rot} F = -x$ and split A in

   $$A_1 = \{(x,y) \in \mathbb{R}^2 : x^2 - 4 < y < 2\}$$
   $$A_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$$

   it follows

   $$\iint_A \operatorname{rot} F \; dxdy = \iint_{A_1} \operatorname{rot} F \; dxdy + \iint_{A_2} \operatorname{rot} F \; dxdy.$$

   We get

   $$-\iint_{A_2} \operatorname{rot} F \; dxdy = \int_0^1 \int_0^{2\pi} (r\cos\theta)\, r \; drd\theta = 0$$

and

$$\iint_{A_1} \operatorname{rot} F \; dxdy = \int_{-\sqrt{6}}^{\sqrt{6}} \int_{x^2-4}^{2} (-x)\, dxdy$$

$$= \int_{-\sqrt{6}}^{\sqrt{6}} -x\left(6-x^2\right) dxdy = \left[-3x^2 + \frac{x^4}{4}\right]_{-\sqrt{6}}^{\sqrt{6}} = 0.$$

So $\iint_A \operatorname{rot} F \; dxdy = 0$.
Step 2. For the calculation of $\int_{\partial A} F \cdot dl$ we define the boundary of A as the union of the curves (see picture for better visualization)

$$\Gamma_0 = \left\{ \alpha(t) = (-t, 2)\, t \in \left(-\sqrt{6}, \sqrt{6}\right) \right\}$$

$$\Gamma_1 \left\{ \beta(t) = \left(t, t^2 - 4\right) t \in \left(-\sqrt{6}, \sqrt{6}\right) \right\}$$

$$\Gamma_2 = \{ \gamma(t) = (\cos t, \sin t)\, t \in (0, 2\pi) \}.$$

We proceed with the calculation

$$\int_{\Gamma_0} F \cdot dl = \int_{-\sqrt{6}}^{\sqrt{6}} (-2t, t) \cdot (-1, 0)\, dt = [t^2]_{-\sqrt{6}}^{\sqrt{6}} = 0$$

$$\int_{\Gamma_1} F \cdot dl = \int_{-\sqrt{6}}^{\sqrt{6}} \left(t\left(t^2-4\right), t^2-4\right) \cdot (1, 2t)\, dt = \int_{-\sqrt{6}}^{\sqrt{6}} 3t\left(t^2-4\right) dt = 0$$

$$\int_{-\Gamma_2} F \cdot dl = -\int_0^{2\pi} (\cos t \sin t, \sin t) \cdot (-\sin t, \cos t)\, dt = 0.$$

so the final result is

$$\int_{\partial A} F \cdot dl = \int_{\Gamma_0} F \cdot dl + \int_{\Gamma_1} F \cdot dl + \int_{-\Gamma_2} F \cdot dl = 0.$$

Exercise 3 (Ex 4.3 page 42)**.**
Before carrying on with the exercise, it is useful to observe that

$$\Omega = \{(x, y) \in \mathbb{R}^2, 0 \le x \le 1 \text{ and } 0 \le y \le 1 - x\}$$

.

1. We find $\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^x$, which leads to

$$\iint_\Omega \Delta f \; dxdy = \int_0^1 \int_0^{1-x} e^x \; dxdy = [(2-x)e^x)]_0^1 = e - 2.$$

2. The calculation can be carried in two ways.

   (a) *Using the Divergence theorem:* we find

$$\int_{\partial\Omega} (\nabla f) \cdot \nu \; dl = \int_{\Omega} \nabla \cdot (\nabla f) = \int_{\Omega} \Delta f \; dxdy = e - 2.$$

   (b) *Explicit calculation:* observe $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ with

$$\Gamma_0 = \{\gamma_0(t) = (t, 0), t \in [0,1]\} \implies \| \gamma_0 \|^2 = 1 \implies \nu = (0, -1)$$

$$\Gamma_1 = \{\gamma_1(t) = (t, 1-t), t \in [0,1]\} \implies \| \gamma_1 \|^2 = 2 \implies \nu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\Gamma_2 = \{\gamma_2(t) = (0, 1-t), t \in [0,1]\} \implies \| \gamma_2 \|^2 = 1 \implies \nu = (-1, 0).$$

   We calculate $\nabla f = (1, e^x)$ and make the observation $\nabla f \cdot \nu = \partial f / \partial \nu$. It follows that

$$\int_{\Gamma_0} \frac{\partial f}{\partial \nu} dl = \int_0^1 \left(e^t, 1\right) \cdot (0, -1) \, dt = -1$$

$$\int_{\Gamma_1} \frac{\partial f}{\partial \nu} dl = \int_0^1 \left(e^{1-t}, 1\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \sqrt{2} dt = e$$

$$\int_{\Gamma_2} \frac{\partial f}{\partial \nu} dl = \int_0^1 (1, 1) \cdot (-1, 0) \, dt = -1$$

$$\implies \int_{\partial\Omega} (\nabla f) \cdot \nu \; dl = e - 2.$$

Exercise 4 (Ex 4.9i page 43)**.**
After calculating $\operatorname{rot} F = 2$, $\operatorname{rot} G_1 = \operatorname{rot} G_2 = 1$, we can use Green's theorem to show that

$$\begin{aligned} \text{Area}\,(\Omega) &= \iint_{\Omega} dxdy = \iint_{\Omega} \frac{1}{2} \operatorname{rot} F \; dxdy = \frac{1}{2} \int_{\partial\Omega} F \cdot dl \\ &= \iint_{\Omega} \operatorname{rot} G_1 \; dxdy = \int_{\partial\Omega} G_1 \cdot dl = \int_{\partial\Omega} G_2 \cdot dl. \end{aligned}$$