EPFL - Fall 2024	Veronica Montanaro
Analysis III - 203 (a)	**Correction**
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Exercise 1 (Ex 2.1 page 17)**.**

1. A parametrization of Γ is given by:

$$x = t \text{ and } y = f(t)$$

i.e.,

$$\gamma(t) = (t, f(t)) \implies \gamma'(t) = (1, f(t))$$

from this, one can obtain

$$\text{length}(\Gamma) = \int_a^b \sqrt{1 + (f'(t))^2}\, \mathrm{d}t.$$

2. We now have $x = t,\ f(t) = \cosh x$ and therefore we can make use of

$$\sqrt{1 + (f'(t))^2} = \cosh t.$$

So we find:

$$\text{length}(\Gamma) = \int_0^1 \cosh t \mathrm{d}t$$
$$= \sinh 1 - \sinh 0 = \frac{e - e^{-1}}{2}$$

3. From $\gamma(t) = (r(t)\cos t, r(t)\sin t)$ we can derive

$$\gamma'(t) = (r'(t)\cos t - r(t)\sin t, r'(t)\sin(t) + r(t)\cos t)$$
$$\implies ||\gamma'(t)||^2 = (r'(t))^2 + (r(t))^2,$$

and therefore

$$\text{length}(\Gamma) = \int_a^b \sqrt{(r'(t))^2 + (r(t))^2}\, \mathrm{d}t.$$

Exercise 2 (Ex 2.4 page 18)**.**
We can write the following parametrization:

$$\gamma'(t) = (-\sin t, \cos t, t), \implies ||\gamma'(t)|| = \sqrt{1 + t^2} \text{ and } f(\gamma(t)) = 1 + t.$$

Therefore

$$\begin{aligned}\int_\Gamma f\,\mathrm{d}l = \int_0^1 (1+t)\sqrt{1+t^2}\,\mathrm{d}t = \int_0^1 \sqrt{1+t^2}\,\mathrm{d}t + \int_0^1 t(\sqrt{1+t^2})\,\mathrm{d}t \\ = \left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left(t+\sqrt{t^2+1}\right) + \frac{1}{3}(1+t^2)\sqrt{1+t^2}\right]_0^1 \\ = -\frac{1}{3} + \frac{7\sqrt{2}}{6} + \frac{1}{2}\ln\left(1+\sqrt{2}\right).\end{aligned}$$

Exercise 3 (Ex 2.2 page 17)**.**

1. With $\gamma_1'(t) = (1,1) \implies F(\gamma_1(t)) = (t^2, t^2 - t)$ we can easily evaluate

$$\int_{\Gamma_1} F \cdot \mathrm{d}l = \int_0^1 (t^2, t^2-t)\cdot(1,1)\mathrm{d}t = \int_0^1 2t^2 - t\mathrm{d}t = \frac{1}{6}$$

2. $\gamma_2'(t) = (1, e^t) \implies F(\gamma_2(t)) = (te^t, e^{2t} - t)$

$$\int_{\Gamma_2} F \cdot \mathrm{d}l = \int_0^1 (te^t, e^{2t}-t)\cdot(1,e^t)\mathrm{d}t = \frac{1}{3}(e^3-1)$$

3. $\gamma_3'(t) = \left(\frac{1}{2\sqrt{t}}, 2t\right) \implies F(\gamma_3(t)) = (t^2\sqrt{t}, t^4 - \sqrt{t})$

$$\int_{\Gamma_3} F \cdot \mathrm{d}l = \int_1^2 (t^2\sqrt{t}, t^4-\sqrt{t})\cdot\left(\frac{1}{2\sqrt{t}}, 2t\right)\mathrm{d}t = \int_1^2 \frac{1}{2}t^2 + 2t^5 - 2t\sqrt{t}\mathrm{d}t = \frac{689}{30} - \frac{16}{5}\sqrt{2}$$

Exercise 4 (Ex 2.3 page 17)**.**

1. For $t \in [0, 2\pi]$, we choose the following parametrization:

$$\gamma(t) = (\cos t, \sin t, 0) \implies \gamma'(t) = (-\sin t, \cos t, 0).$$

   And proceed to calculate:

$$\int_\Gamma F\cdot\mathrm{d}l = \int_0^{2\pi} (\cos t, 0, \sin t)\cdot(-\sin t, \cos t, 0)\,\mathrm{d}t = -\int_0^{2\pi} \cos t \sin t\,\mathrm{d}t = 0$$

2. $t \in [0,1],\ \gamma(t) = (t, e^t, t) \implies \gamma'(t) = (1, e^t, 1)$.

$$\int_\Gamma F \cdot \mathrm{d}l = \int_0^1 (t, e^t, t) \cdot (1, e^t, 1)\,\mathrm{d}t = \int_0^1 (2t + e^{2t})\,\mathrm{d}t = \frac{1+e^2}{2}.$$

Exercise 5 (Ex 2.6 page 18)**.**

1. First, we need to observe that Γ is given for $y = 2x\sqrt{(1-x^2)}$. We can then show that by choosing $x = \sin t, t \in [0, \pi]$, we indeed get $y = 2\sin t \cos t = \sin(2t)$. Therefore, $\gamma(t)$ is a valid parametrization for Γ.

2.

$$\begin{aligned}\int_\Gamma F \cdot \mathrm{d}l &= \int_0^\pi (\sin t + \sin(2t), -\sin(t)) \cdot (\cos t, 2\cos(2t))\,\mathrm{d}t \\ &= \int_0^\pi (\sin t \cos t + \sin(2t)\cos t - 2\sin t \cos(2t))\,\mathrm{d}t = \frac{8}{3}.\end{aligned}$$

Exercise 6 (Ex 2.5 page 18)**.**
Let $\gamma : [a,b] \to \Gamma$ be a parametrization of Γ, with $\gamma(a) = A, \gamma(b) = B$. Moreover, $F(\gamma(t)) = m\gamma''(t)$. Then we can find

$$\begin{aligned}\int_\Gamma F \cdot \mathrm{d}l = \int_a^b m\gamma''(t)\dot{\gamma}(t)\,\mathrm{d}t &= m\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}||\gamma'(t)^2||\right]\mathrm{d}t \\ &= \frac{m}{2}(||\gamma'(b)^2|| - ||\gamma'(a)^2||).\end{aligned}$$

Does this formula ring a bell? :)