EPFL - automne 2024	Leonid Monin
Analyse III	**Corrigé**
Série 1	12 septembre 2024

Exercice 1.
On a

$$\begin{aligned}
\frac{\partial}{\partial x_i}(F\cdot G) =& \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n F_jG_j\right)\\
=& \sum_{j=1}^n \frac{\partial}{\partial x_i}\left(F_jG_j\right)\\
=& \sum_{j=1}^n \left(\frac{\partial F_j}{\partial x_i}G_j + F_j\frac{\partial G_j}{\partial x_i}\right)\\
=& \sum_{j=1}^n \frac{\partial F_j}{\partial x_i}G_j + \sum_{j=1}^n F_j\frac{\partial G_j}{\partial x_i}\\
=& \left\langle \frac{\partial F}{\partial x_i}, G\right\rangle + \left\langle F, \frac{\partial G}{\partial x_i}\right\rangle
\end{aligned}$$

Exercice 2. (i) On calcule

$$\operatorname{Jac}(u)(r,\theta) = \det \nabla u(r,\theta) = \det\begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta)\end{pmatrix} = r.$$

(ii) On calcule

$$\begin{aligned}
\operatorname{Jac}(u)(r,\theta,\varphi) =& \det\nabla u(r,\theta,\varphi)\\
=& \det\begin{pmatrix} \sin(\theta)\cos(\varphi) & r\cos(\theta)\cos(\varphi) & -r\sin(\theta)\sin(\varphi)\\ \sin(\theta)\sin(\varphi) & r\cos(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) \\ \cos(\theta) & -r\sin(\theta) & 0\end{pmatrix}\\
=& r^2\sin(\theta).
\end{aligned}$$

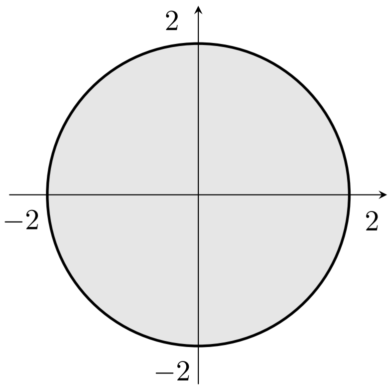
(iii) On calcule

$$\operatorname{Jac}(u)(r,\theta,z) = \det\nabla u(r,\theta,z) = \det\begin{pmatrix} \cos(\theta) & -r\sin(\theta) & 0\\ \sin(\theta) & r\cos(\theta) & 0\\ 0 & 0 & 1\end{pmatrix} = r.$$

(iv) On calcule

$$\operatorname{Jac}(u)(x,y,z) = \det\nabla u(x,y,z) = \det\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = 1.$$

Exercice 3. (i)



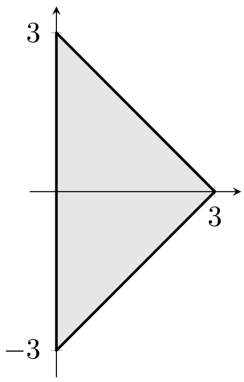
On utilise les coordonnées polaires :

$$A = \{(r\cos\theta, r\sin\theta) \mid r \in [0,2], \theta \in [0, 2\pi]\}$$

et on calcule

$$\begin{aligned}\int_A f(y)dy =& \int_0^2 \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r d\theta dr\\ =& \int_0^2 \int_0^{2\pi} \frac{1}{r} r d\theta dr\\ =& 4\pi\end{aligned}$$

(ii)



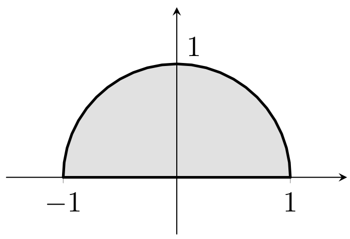
On utilise les coordonnées cartésiennes et on calcule

$$\begin{aligned}\int_A f(y)dy =& \int_0^3 \int_{x-3}^{3-x} f(x,y) dydx\\ =& \int_0^3 \int_{x-3}^{3-x} x^2 + \sin^3(y) dydx\\ =& \int_0^3 (6-2x)x^2 dx + \int_0^3 \underbrace{\int_{x-3}^{3-x} \sin^3(y) dy}_{=0} \, dx.\end{aligned}$$

Où la deuxième intégrale est nulle car on intègre une fonction impaire sur un intervalle symétrique centré en 0. Ainsi,

$$\begin{aligned}
\int_A f(y)dy &= \int_0^3 6x^2 - 2x^3 dx \\
&= \left[2x^3 - \frac{1}{2}x^4\right]_{x=0}^{x=3} \\
&= \frac{27}{2}
\end{aligned}$$

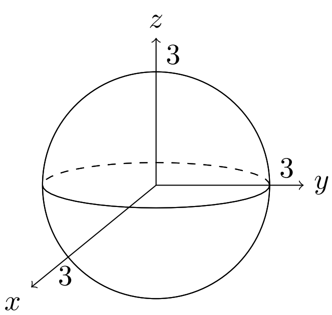
(iii)



On utilise les coordonnées cartésiennes. On pourrait aussi utiliser les coordonnées polaires, auquel cas il faut faire attention à ce que l'angle θ ne parcourt que $[0, \pi]$

$$\begin{aligned}
\int_A f(y)dy &= \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} y(1+x^2)dydx \\
&= \int_{-1}^{1} (1+x^2) \left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{1-x^2}} dx \\
&= \frac{1}{2} \int_{-1}^{1} (1+x^2)(1-x^2)dx \\
&= \frac{1}{2} \int_{-1}^{1} -x^4 + 1dx \\
&= \frac{1}{2} \left[-\frac{1}{5}x^5 + x\right]_{x=-1}^{x=1} = \frac{4}{5}.
\end{aligned}$$

(iv)



On utilise les coordonnées sphériques :

$$A = \{(r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi) \mid r \in [0,3], \theta \in [0, 2\pi], \varphi \in [0, \pi]\}$$

et on calcule

$$\begin{aligned}\int_A f(y)dy &= \int_0^3 \int_0^{2\pi} \int_0^{\pi} r \sin\varphi r^2 \sin\varphi d\varphi d\theta dr \\ &=2\pi \int_0^3 r^3 dr \int_0^{\pi} \sin^2\varphi d\varphi\end{aligned}$$

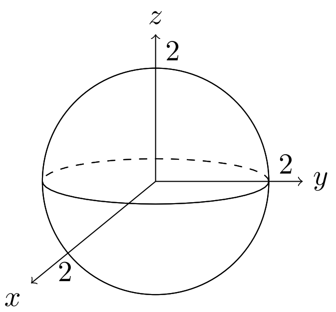
On rappelle que

$$\frac{d}{dt}\left[\frac{1}{2}\left(t-\sin(t)\cos(t)\right)\right] = \sin^2(t).$$

D'où,

$$\begin{aligned}\int_A f(y)dy &==2\pi \left[\frac{1}{4}r^4\right]_{r=0}^{r=3} \left[\frac{1}{2}\left(\varphi-\sin(\varphi)\cos(\varphi)\right)\right]_{\varphi=0}^{\varphi=\pi} \\ &=\frac{81}{4}\pi^2.\end{aligned}$$

(v)



On utilise les coordonnées sphériques :

$$A = \{(r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi) \mid r\in[0,2], \theta\in[0,2\pi], \varphi\in[0,\pi]\}$$

et on calcule

$$\begin{aligned}\int_A f(y)dy &= \int_0^2 \int_0^{2\pi} \int_0^{\pi} \arccos(\cos\varphi) r^2 \sin\varphi d\varphi d\theta dr \\ &=2\pi \int_0^2 r^2 dr \int_0^{\pi} \varphi \sin\varphi d\varphi \\ &\overset{\text{IPP}}{=} 2\pi \left[\frac{1}{3}r^3\right]_{r=0}^{r=2} \left([\varphi(-\cos\varphi)]_{\varphi=0}^{\varphi=\pi} + \int_0^{\pi} \cos\varphi d\varphi \right) \\ &=\frac{16}{3}\pi^2\end{aligned}$$

Exercice 4. (i) Pour tout $1 \le i \le n$ et $x \neq 0$,

$$\frac{\partial f(x)}{\partial x_i} = \frac{\partial}{\partial x_i}\sqrt{x_1^2+\cdots+x_n^2} = \frac{1}{2\sqrt{x_1^2+\cdots+x_n^2}} \cdot 2x_i = \frac{x_i}{||x||}.$$

(ii) En utilisant le point (i),

$$\frac{\partial g(x)}{\partial x_i} = \frac{\partial}{\partial x_i}||x||^p = p||x||^{p-1} \cdot \frac{\partial}{\partial x_i}||x|| = p||x||^{p-1} \cdot \frac{x_i}{||x||} = p||x||^{p-2}x_i.$$

Exercice 5.
Soit une fonction $f = f(x) : \mathbb{R}^3 \to \mathbb{R}$ de classe C_1, et la fonction $g = g(t) : \mathbb{R} \to \mathbb{R}$ définie par

$$g(t) = f(t, t^2, t + t^2).$$

(i) Par la formule du théorème 2, nous avons directement

$$\begin{aligned} g'(t) &= (t)' \cdot \frac{\partial f(t, t^2, t+t^2)}{\partial x_1} + (t^2)' \cdot \frac{\partial f(t, t^2, t+t^2)}{\partial x_2} + (t+t^2)' \cdot \frac{\partial f(t, t^2, t+t^2)}{\partial x_3} \\ &= \frac{\partial f(t, t^2, t+t^2)}{\partial x_1} + 2t \cdot \frac{\partial f(t, t^2, t+t^2)}{\partial x_2} + (2t+1) \cdot \frac{\partial f(t, t^2, t+t^2)}{\partial x_3} \end{aligned}$$

(ii) En notant $f(x) = f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$, nous avons

$$\frac{\partial f(x)}{\partial x_i} = 2x_i, \quad i = 1, 2, 3.$$

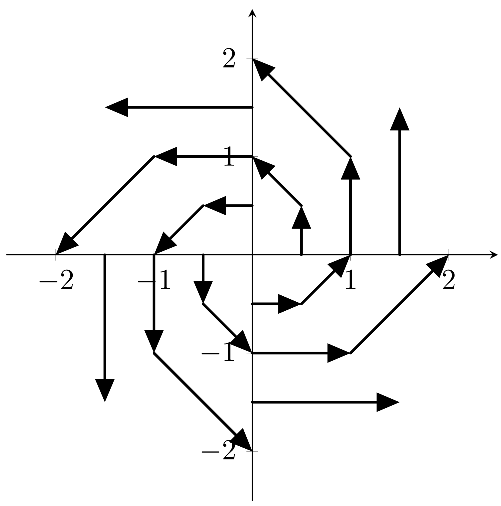
Donc par ce qui précède

$$g'(t) = 2t + 2t \cdot 2t^2 + (2t+1) \cdot 2(t + t^2) = 8t^3 + 6t^2 + 4t.$$

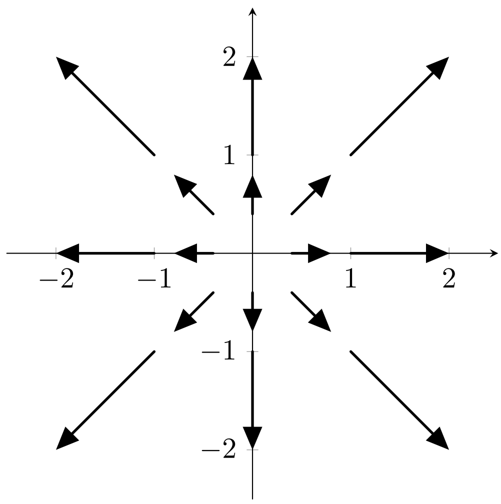
Nous arrivons également au même résultat si nous développons explicitement

$$g(t) = t^2 + (t^2)^2 + (t + t^2)^2 = 2t^4 + 2t^3 + 2t^2.$$

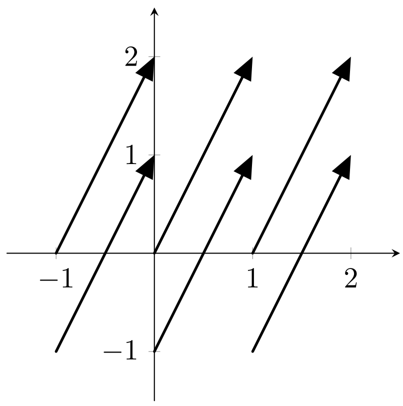
Exercice 6. (i) $\operatorname{div} F(x) = 0, \quad \operatorname{rot} F(x) = 2.$



(ii) $\operatorname{div} G(x) = 2, \quad \operatorname{rot} G(x) = 0.$



(iii) $\operatorname{div} H(x) = 0, \quad \operatorname{rot} H(x) = 0.$



(iv) $\operatorname{div} I(x) = 0, \quad \operatorname{rot} I(x) = -1.$