08/10/2024

CHAPTER 3: SURFACE INTEGRALS, DIVERGENCE THEOREM AND STOKES THEOREM

3.1 Recalls and preliminary notations

3.1.1 Change of variables in an integral with several variables

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous bounded function defined by $f(x) = f(x_1, x_2, \dots, x_n)$. Let $\Omega \subset \mathbb{R}^n$ an open domain.

A change of variables is described by a bijective function u defined by:

$$u : \Omega \longrightarrow u(\Omega) \subset \mathbb{R}^n$$
$$(y_1, y_2, \dots y_n) \longmapsto u(y_1, \dots, y_n) = (x_1, \dots, x_n)$$

we express the old variables $(x_1, \dots, x_2)$ as a function of the new ones $(y_1, \dots, y_n)$

$$x_1 = u_1(y_1, \dots, y_n)$$

$$x_2 = u_2(y_1, \dots, y_n)$$

$$\dots$$

$$x_n = u_n(y_1, \dots, y_n)$$

u is defined such that:

$$\underbrace{\det \nabla u(y)}_{\text{Jacobian}} = \det \begin{pmatrix} \frac{\partial u_1}{\partial y_1} & \dots & \frac{\partial u_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial u_n}{\partial y_1} & \dots & \frac{\partial u_n}{\partial y_n} \end{pmatrix} \neq 0 \quad \forall y \in \Omega$$

The formula for changing variables in an integral is:

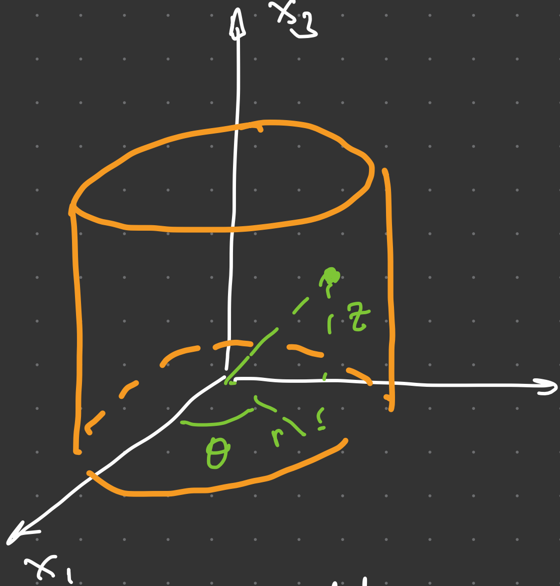
$$\iint \cdots \int_{\Omega} f(x_1, \dots, x_n)\, dx_1\, dx_2, \dots dx_n$$

$$= \iint \cdots \int_{u^{-1}(\Omega)} f(u(y_1, \dots, y_n))\, \| \det \nabla u(y) \|\, dy_1\, dy_2 \dots dy_n$$

Example 1: cylindrical coordinates

old variables (x_1, x_2, x_3): Cartesian coords. in $\mathbb{R}^3$

New variables $(y_1, y_2, y_3) = (r, \theta, z)$: cylindrical coords in $\mathbb{R}^3$



$$x_1 = u_1(r,\theta,z) = r\cos\theta \qquad r > 0$$

$$x_2 = u_2(r,\theta,z) = r\sin\theta \qquad \theta \in [0, 2\pi]$$

$$x_3 = u_3(r,\theta,z) = z$$

$$\det \nabla u(r,\theta,z) = \det \begin{pmatrix} \frac{\partial u_1}{\partial r} & \frac{\partial u_1}{\partial \theta} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial r} & \frac{\partial u_2}{\partial \theta} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial r} & \frac{\partial u_3}{\partial \theta} & \frac{\partial u_3}{\partial z} \end{pmatrix} =$$

$$= \det \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r \longrightarrow |\det \nabla u(r,\theta,z)| = r$$

$r > 0$

$$\iiint_\Omega f(x_1,x_2,x_3)\,dx_1dx_2dx_3 = \iiint_{u^{-1}(\Omega)} f(r\cos\theta, r\sin\theta, z)\; r\,dr\,d\theta\,dz$$

If $f=1$, $\Omega = \{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1^2+x_2^2 \le R^2 \text{ and } x_3\in[0,h]\}$

$$\iiint_\Omega f\,dx_1dx_2dx_3 = \int_0^R r\,dr \int_0^{2\pi} d\theta \int_0^h dz = h\pi R^2$$

Example 2:

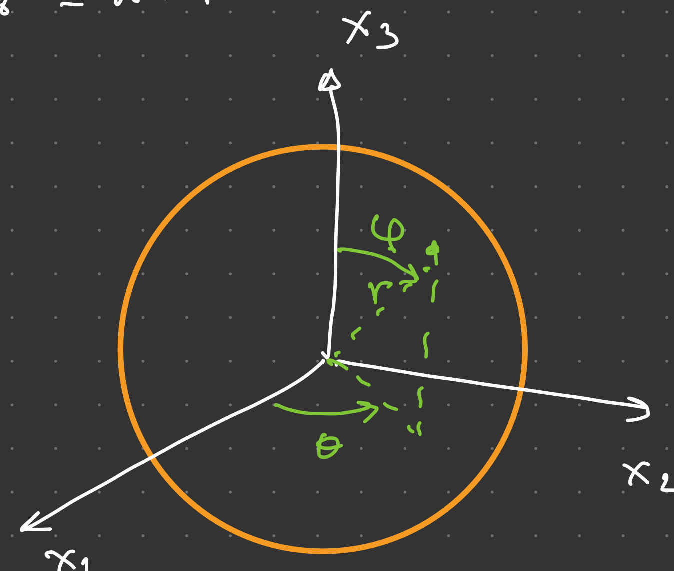
Old variables (x_1,x_2,x_3)

New variables (r,θ,φ)

$x_1 = r\sin\varphi\cos\theta$

$x_2 = r\sin\varphi\sin\theta$

$x_3 = r\cos\varphi$



$$\det \nabla u(r,\theta,\varphi) = -r^2 \sin\varphi$$

$$|\det \nabla u(r,\theta,\varphi)| = r^2 \sin\varphi$$

$r>0,\ \varphi \in [0,\pi]$

$$\iiint_\Omega f\, dx_1\, dx_2\, dx_3 = \ldots \iiint_{u^{-1}(\Omega)} f(u(r,\theta,\varphi))\, r^2 \sin\varphi\, dr\, d\varphi\, d\theta$$

$f=1,\ \Omega = \{(x_1,x_2,x_3)\in \mathbb{R}^3 : x_1^2+x_2^2+x_3^2 < R^2\}$ then

$$\iiint_\Omega f\, dx_1\, dx_2\, dx_3 = \int_0^R r^2\, dr \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi = \frac{4}{3}\pi R^3$$

3.1.2 Surfaces in $\mathbb{R}^3$:

- New notations

a) for a function $f : \mathbb{R}^2 \to \mathbb{R}^3$ we denote

$$f(x,y) = \left(f^1(x,y), f^2(x,y), f^3(x,y) \right)$$

b) for a function $g(x,y)$, we denote

$$\frac{\partial g}{\partial x} = g_x \quad , \quad \frac{\partial g}{\partial y} = g_y \quad , \quad \frac{\partial^2 g}{\partial x \partial y} = g_{xy}$$

- Definition 1:

$\Sigma \subset \mathbb{R}^3$ is called a <u>regular surface</u> if:

a) There exists $A \subset \mathbb{R}^2$, a bounded open domain, s.t., ∂A

is a (piecewise) closed regular simple curve.

b) There exists a function $\sigma : \bar{A} \longrightarrow \mathbb{R}^3$

$$(u,v) \mapsto \sigma(u,v) = (\sigma^1(u,v), \sigma^2(u,v), \sigma^3(u,v))$$

that satisfies:

- $\sigma \in C^1(\bar{A}, \mathbb{R}^3)$; $\sigma(\bar{A}) = \Sigma$ is injective over A.
- the vector $\sigma_u \times \sigma_v = \begin{vmatrix} e_1 & e_2 & e_3 \\ \sigma^1_u & \sigma^2_u & \sigma^3_u \\ \sigma^1_v & \sigma^2_v & \sigma^3_v \end{vmatrix}$

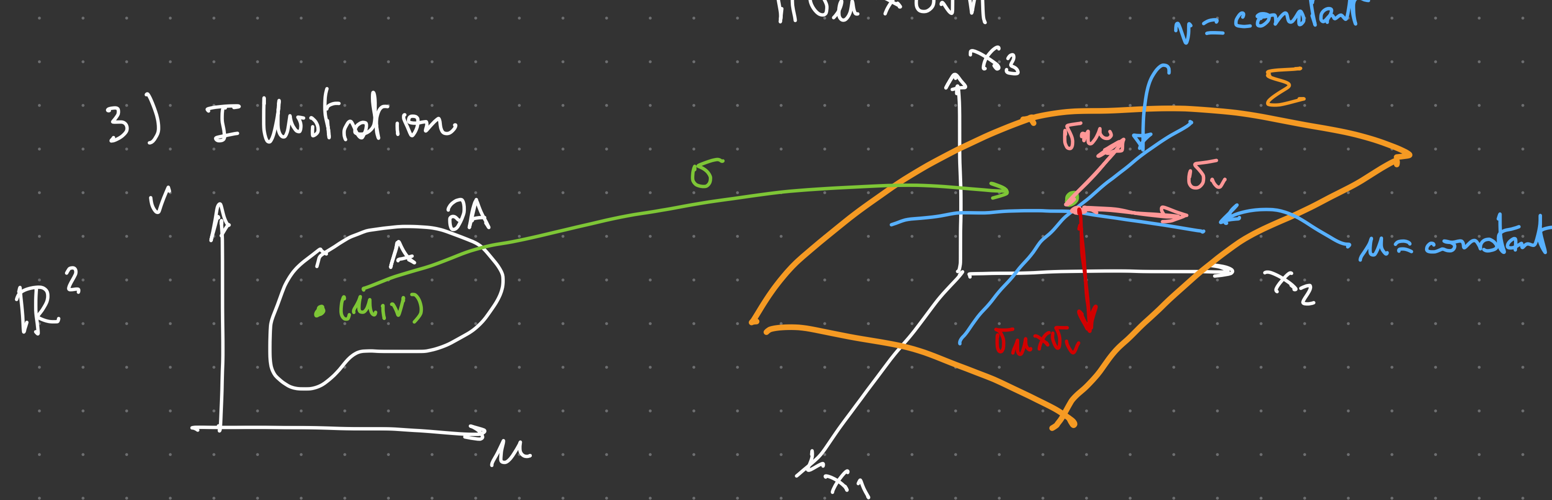
is s.t. $\|\sigma_u \times \sigma_v\| \neq 0 \quad \forall (u,v) \in A$.

• Remarks:

1) σ is a parameterization of Σ

2) $\sigma_u \times \sigma_v$ $\left(\frac{\partial \sigma}{\partial u} = \sigma_u, \ \frac{\partial \sigma}{\partial v} = \sigma_v \right)$ is the normal to the surface. $\nu(u,v) = \dfrac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$ is the unit normal.

3) Illustration



4) Analogy with curves in $\mathbb{R}^3$

curve Γ	surface Σ
• $\gamma : [a,b] \subset \mathbb{R} \longrightarrow \Gamma \subset \mathbb{R}^3$ $t \longmapsto \gamma(t)$	• $\sigma : \bar{A} \subset \mathbb{R}^2 \longrightarrow \Sigma \subset \mathbb{R}^3$ $(u,v) \longmapsto \sigma(u,v)$
• $\gamma([a,b]) = \Gamma$ and injective (over $[a,b[$)	• $\sigma(\bar{A}) = \Sigma$ and injective over A.
• $\lVert\gamma'(t)\rVert \neq 0$ (non-null tangent vector)	• $\lVert\sigma_u \times \sigma_v\rVert \neq 0$ (non-null normal vector).
• $dl = \lVert\gamma'(t)\rVert\, dt$	• $dA = \lVert\sigma_u \times \sigma_v\rVert\, du\, dv$

- Definition 2:

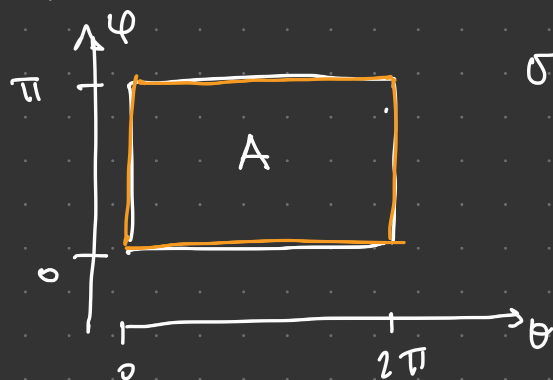
we say that $\Sigma \subset \mathbb{R}^3$ is a <u>piecewise</u> regular surface if

$\exists$ regular surfaces $\Sigma_1, \Sigma_2, \dots, \Sigma_k$, s.t. $\Sigma = \bigcup_{i=1}^{k} \Sigma_i$.

- Example 1: sphere of radius R (orange peel)

$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = R^2\}$$

Parameterization: we define $A =]0, 2\pi[\times]0, \pi[$



$$\sigma : \bar{A} \longrightarrow \Sigma$$

$$(\theta, \varphi) \longmapsto \sigma(\theta,\varphi) = (\sigma^1(\theta,\varphi), \sigma^2(\theta,\varphi), \sigma^3(\theta,\varphi)) = (R\sin\varphi\cos\theta, R\sin\varphi\sin\theta, R\cos\varphi)$$

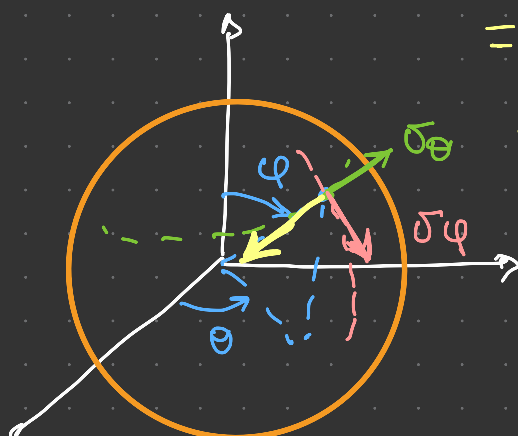
Normal $\sigma_\theta \times \sigma_\varphi = \begin{vmatrix} e_1 & e_2 & e_3 \\ \sigma_\theta^1 & \sigma_\theta^2 & \sigma_\theta^3 \\ \sigma_\varphi^1 & \sigma_\varphi^2 & \sigma_\varphi^3 \end{vmatrix} =$

In yellow, correction after lecture.

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ -R\sin\varphi\sin\theta & R\sin\varphi\cos\theta & 0 \\ R\cos\varphi\cos\theta & R\cos\varphi\sin\theta & -R\sin\varphi \end{vmatrix} =$$

$$= -(R^2\sin^2\varphi\cos\theta, R^2\sin^2\varphi\sin\theta, R^2\sin\varphi\cos\varphi)$$

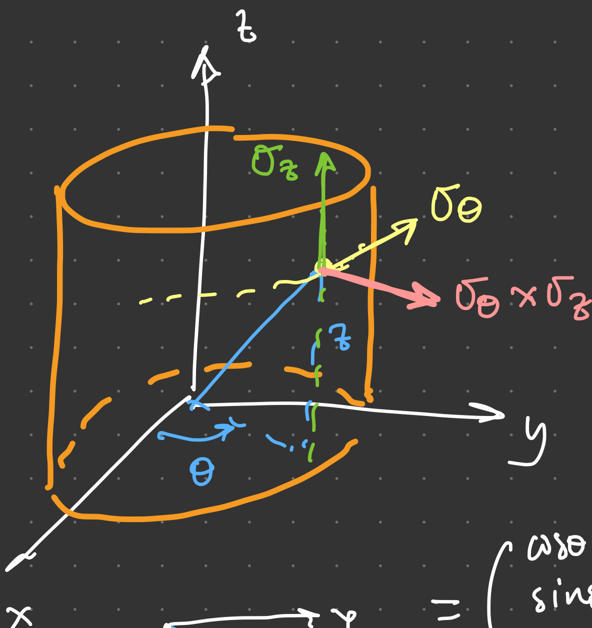
$$= -R\sin\varphi\,\underbrace{(R\sin\varphi\cos\theta, R\sin\varphi\sin\theta, R\cos\varphi)}_{\sigma(\theta,\varphi)}$$



$$= -R\sin\varphi\,\sigma(\theta,\varphi)$$

- Example 2: Cylinder $(R=1,\ h=1)$

$$\Sigma = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2=1 \text{ and } 0 \le z \le 1 \}$$

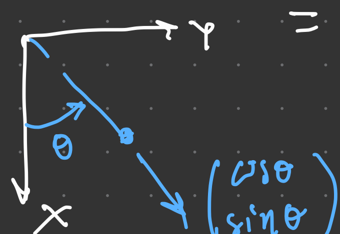


$$A =]0, 2\pi[\times]0,1[$$

$$\sigma : \bar{A} \longrightarrow \Sigma$$

$$(\theta, z) \longmapsto \sigma(\theta,z) = (\cos\theta, \sin\theta, z)$$

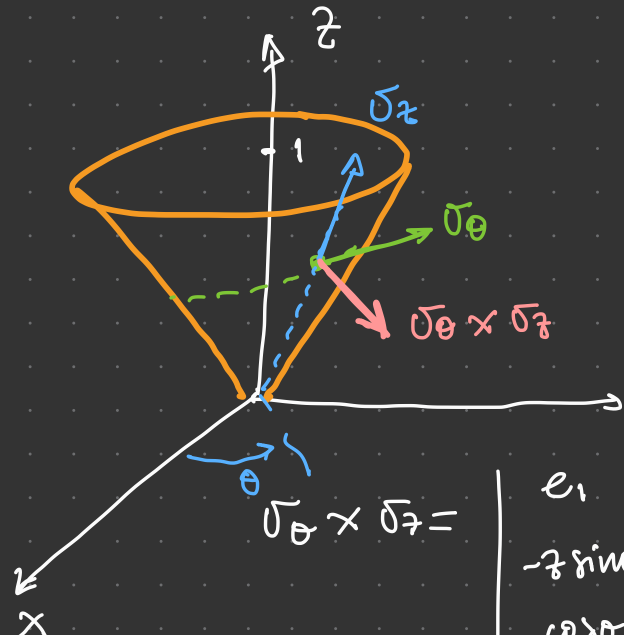
$$\sigma_\theta \times \sigma_z = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$



$$= \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$\sigma_\theta \times \sigma_z$ is external normal.

Example 3: Cone of angle $\pi/2$



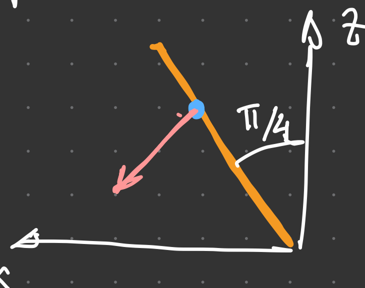
$$\Sigma = \{ (x, y, z) \in \mathbb{R}^3 \quad z^2 = x^2 + y^2 \text{ and } 0 \le z \le 1 \}.$$

$$A =]0, 2\pi[\times]0, 1[$$

$$\sigma : \bar{A} \longrightarrow \Sigma$$
$$(\theta, z) \longmapsto (z\cos\theta, z\sin\theta, z)$$

$$\sigma_\theta \times \sigma_z = \begin{vmatrix} e_1 & e_2 & e_3 \\ -z\sin\theta & z\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = \begin{pmatrix} z\cos\theta \\ z\sin\theta \\ -z \end{pmatrix}$$

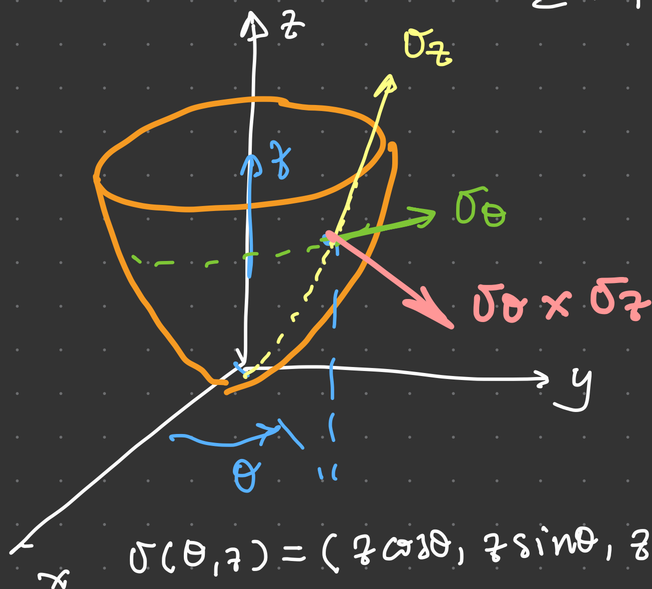
$\theta = 0$



$$\sigma_\theta \times \sigma_z \, (\theta = 0) = \begin{pmatrix} z \\ 0 \\ -z \end{pmatrix}$$

- Example 4: symmetric paraboloid (glass of wine)

$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2 \text{ and } 0 \le z \le 1\}$$



$$\sigma(\theta, z) = (z\cos\theta, z\sin\theta, z^2)$$

$x=0 \longrightarrow z = y^2$