CHAPTER 4 : Fourier series

4.1 Introduction

4.1.1 Motivation

- Taylor series provides a representation of a function : $f : \mathbb{R} \to \mathbb{R}$ infinitely differentiable as a sum of monomials:

$$f(x) = \sum_{n=0}^{\infty} \alpha_n x^n \quad \text{where} \quad \alpha_n = \frac{f^{(n)}(0)}{n!}$$ are the coefficients of Taylor around $x=0$.

Example: $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x \in \mathbb{R}$

- Fourier series : approximate a periodic function as an infinite sum of sines and cosines.

Problem: given $f : \mathbb{R} \to \mathbb{R}$ a T-periodic, can we write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right) \right)$$

$a_n, b_n \in \mathbb{R}$?

Answer: YES! if we impose certain conditions to f

4.1.2 Recalls and preliminary results

- Recall 1: A function $f: \mathbb{R} \to \mathbb{R}$ is T-periodic if there exist $T > 0$ s.t. $f(x+T) = f(x) \ \forall x \in \mathbb{R}$.

  T is the <u>period</u> of f.

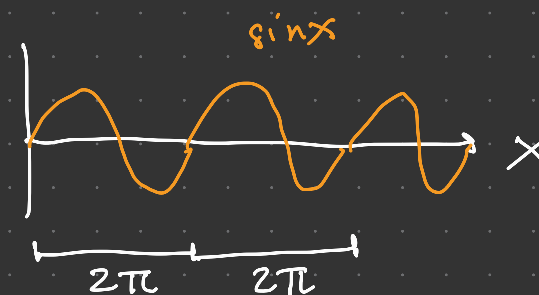
  Examples: $u_n(x) = \cos\left(\frac{2\pi n}{T} x\right)$ or $v_n = \sin\left(\frac{2\pi n}{T} x\right)$

  are $\frac{T}{n}$-periodic for $n \in \mathbb{N}^*$ $\quad (\mathbb{N}^* = \{1, 2, 3, \dots\})$

$$u_n\left(x + \frac{T}{n}\right) = \cos\left(\frac{2\pi n}{T}\left(x + \frac{T}{n}\right)\right) = \cos\left(\frac{2\pi n}{T} x + 2\pi\right)$$

$$= \cos\left(\frac{2\pi n}{T} x\right) = u_n(x)$$

The same for v_n

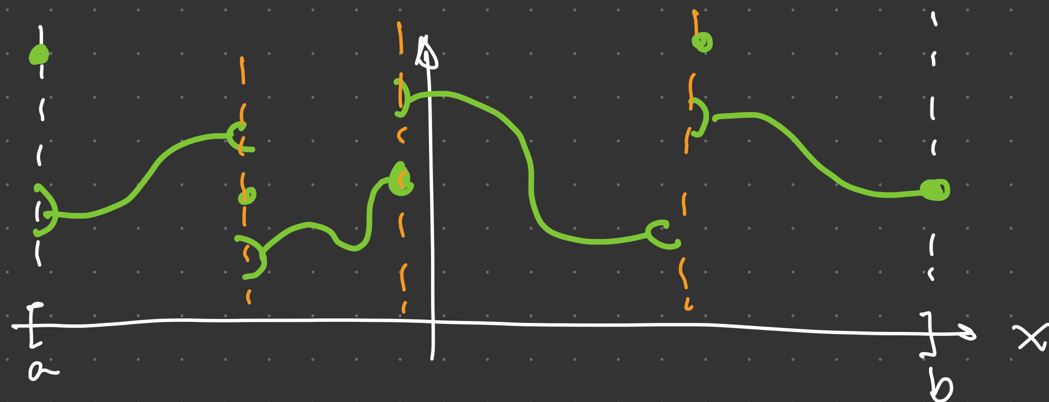


$\sin x$ is $2\pi n$-periodic $n \in \mathbb{N}^*$

• Recall 2: A function $f : \mathbb{R} \to \mathbb{R}$ is piecewise defined (or simply piecewise) if it has a finite number of discontinuities over every bounded interval. and if at each discontinuity x the limits:

$$\lim_{\substack{t \to x \\ t > x}} f(t) \doteq f(x+0) \quad \text{and} \quad \lim_{\substack{t \to x \\ t < x}} f(t) \doteq f(x-0)$$

exist and are bounded (finite).



• Result 1: let $m, n \in \mathbb{N}^*$ and $T > 0$ then:

a) $$\frac{2}{T}\int_0^T \cos\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi m}{T}x\right)dx = \frac{2}{T}\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\sin\left(\frac{2\pi m}{T}x\right)dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

b) $$\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi m}{T}x\right)dx = 0$$

Proof: for the sake of simplicity $T = 2\pi$

a) $$I = \frac{1}{\pi}\int_0^{2\pi} \cos(nx)\cos(mx)\,dx = \frac{1}{2\pi}\int_0^{2\pi} \cos((n-m)x)\,dx + \frac{1}{2\pi}\int_0^{2\pi} \cos((n+m)x)\,dx$$

$$\cos a \cos b = \frac{1}{2}\left(\cos(a-b) + \cos(a+b)\right)$$

If $n \neq m$

$$I = \frac{1}{2\pi} \underbrace{\left.\frac{\sin((n-m)x)}{n-m}\right|_0^{2\pi}}_{=0 \text{ (because } n\neq m)} + \frac{1}{2\pi} \underbrace{\left.\frac{\sin((n+m)x)}{n+m}\right|_0^{2\pi}}_{=0}$$

If $n = m$

$$I = \cancel{\frac{1}{2\pi} \left.\frac{\sin(0x)}{n-m}\right|_0^{2\pi}} + \frac{1}{2\pi} \underbrace{\left.\frac{\sin((n+m)x)}{n+m}\right|_0^{2\pi}}_{=0}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos(0x)\,dx = \frac{1}{2\pi} \int_0^{2\pi} dx = 1.$$

For $I = \frac{1}{2\pi} \int_0^{2\pi} \sin(nx)\sin(mx)\,dx = \ldots$ $\approx$

$\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$

$$= \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

$\sin a \cos b = \frac{1}{2}(\sin(a-b) + \sin(a+b))$

b) $$K = \int_0^{2\pi} \sin(nx) \cos(mx)\, dx = \frac{1}{2} \int_0^{2\pi} \left(\sin((n-m)x) + \sin((n+m)x) \right)$$

If $n \neq m$

$$K = \frac{1}{2} \left(- \underbrace{\frac{\cos((n-m)x)}{n-m} \Bigg|_0^{2\pi}}_{=0} - \underbrace{\frac{\cos((n+m)x)}{n+m} \Bigg|_0^{2\pi}}_{=0} \right) = 0.$$

If $n = m$

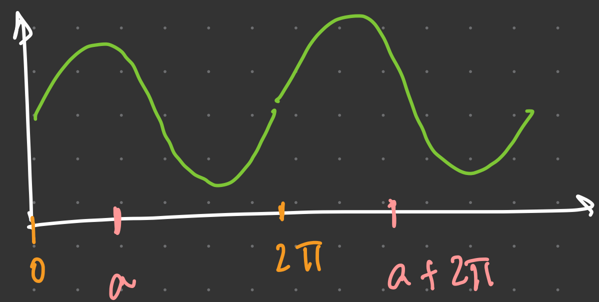
$$K = \frac{1}{2} \int_0^{2\pi} \underbrace{\sin(0)}_{=0}\, dx + \underbrace{\frac{1}{2} \int_0^{2\pi} \sin((n+m)x)\, dx}_{=0} = 0$$

• Result 2: Let $f : \mathbb{R} \to \mathbb{R}$ a T-periodic function

Then:

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx, \qquad \forall a \in \mathbb{R}.$$

Example:



Proof:

$$\int_a^{a+T} f(x)\,dx = \cancel{\int_a^0 f(x)\,dx} + \int_0^T f(x)\,dx + \int_T^{a+T} f(x)\,dx$$

$$\int_T^{a+T} f(x)\,dx \overset{y=x-T}{=} \int_0^a f(y+T)\,dy \overset{f \text{ is } T\text{-periodic}}{=} \int_0^a f(y)\,dy = \cancel{-\int_a^0 f(x)\,dx}.$$

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx.$$

4.2 Fourier series of a periodic function

4.2.1 Definitions:

Let $f: \mathbb{R} \to \mathbb{R}$ be a T-periodic piecewise-defined function

For $n \in \mathbb{N}^*$, the partial Fourier series of f and order N is

$$F_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right)\right)$$

where a_n and b_n are denoted as Fourier coefficients of f.

and are given by

$$a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{2\pi n}{T} x\right) dx \quad n = 0, 1, \ldots, N$$

$$b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi n}{T} x\right) dx \quad n = 1, 2, \ldots, N.$$

We call Fourier series of f to the limit

$$F f(x) = \lim_{N \to \infty} F_N f(x)$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T}\right) x \right)$$

when the limit exists...

What is the relationship between $F f(x)$ and $f(x)$?

4.2.2 Heuristic justification of the definition

For save the simplicity $T = 2\pi$. Then

$$F f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx.$$

- $$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)\right)dx$$

$f(x) = Ff(x)$

$$= \int_0^{2\pi}\frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} a_n\int_0^{2\pi}\cos(nx)\,dx + \sum_{n=1}^{\infty} b_n\int_0^{2\pi}\sin(nx)\,dx$$

$$= \pi a_0 + \sum_{n=1}^{\infty}\left[a_n \underbrace{\frac{\sin(nx)}{n}\Big|_0^{2\pi}}_{=0} - b_n \underbrace{\frac{\cos(nx)}{n}\Big|_0^{2\pi}}_{=0}\right] = \pi a_0$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

• $$\int_0^{2\pi} f(x)\cos(kx)\,dx = \int_0^{2\pi}\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)\right)\cos kx\,dx$$

$k\in\mathbb{N}^*$

$Ff(x)=f(x)$

$$= \frac{a_0}{2}\underbrace{\int_0^{2\pi}\cos kx\,dx}_{=0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_0^{2\pi}\cos nx\cos kx\,dx}_{=\begin{cases}\pi & \text{if } k=n\\ 0 & \text{if } k\neq n\end{cases}} + \sum_{n=1}^{\infty} b_n \underbrace{\int_0^{2\pi}\sin nx\cos kx\,dx}_{=0}.$$

$$= a_k\pi \longrightarrow a_k=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)\,dx \quad \text{for } k\in\mathbb{N}^*$$

• $$\int_0^{2\pi} f(x)\sin kx = \cdots = \pi b_k \longrightarrow b_k=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx$$

for $k\in\mathbb{N}^*$

4.2.3 Fourier series convergence

Questions

- Which is the relationship between $Ff(x)$ and $f(x)$?
- When is it possible to represent a T-periodic function as an infinite sum of $\cos\left(\frac{2\pi n}{T}x\right)$ and $\sin\left(\frac{2\pi n}{T}x\right)$ which are $\frac{T}{n}$-periodic?

Answer: Dirichlet theorem

Let $f: \mathbb{R} \to \mathbb{R}$ be a T-periodic is piecewise defined

then:

$$Ff(x) = \lim_{N \to \infty} F_N f(x) \text{ exists. } \forall x \in \mathbb{R} \text{ and}$$

$$Ff(x) = \frac{1}{2}\left(f(x+0) + f(x-0)\right)$$

In particular, if f is continuous in x, then

$$f(x+0) = f(x-0) = f(x) \text{ and } Ff(x) = f(x)$$

- Comment: for T-periodic function satisfying the hypothesis of Dirichlet theorem we can write

$$\frac{1}{2}\left(f(x+0)+f(x-0)\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n}{T}x\right) + b_n \sin\left(\frac{2\pi n}{T}x\right)\right]$$

or in other words, f can be represented as a superposition of $u_n = \cos\left(\frac{2\pi n}{T}x\right)$ and $v_n = \sin\left(\frac{2\pi n}{T}x\right)$ that constitute a basis.