CHAPTER 2: CURVILINEAR INTEGRALS, CONSERVATIVE FIELDS, AND GREEN'S THEOREM

2.1 Curves in $\mathbb{R}^n$

2.1.1 Definitions

Let $n \geq 1$

- Definition 1: $\Gamma \subset \mathbb{R}^n$ is a <u>regular simple curve</u> if there exists an interval $[a, b]$ and a function

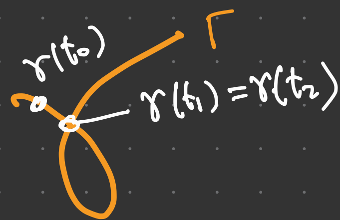
$$\gamma : [a,b] \to \mathbb{R}^n$$
$$t \longmapsto \gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$$

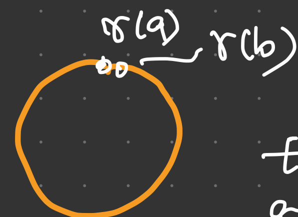
such that:

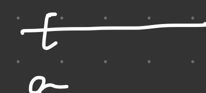
- $\gamma([a,b]) = \Gamma = \{ x \in \mathbb{R}^n : \exists t \in [a,b] \text{ with } x = \gamma(t) \}$

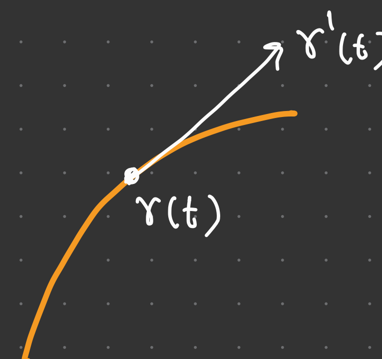
- γ is injective on $[a,b[$: $\forall\, t_1, t_2 \in [a,b[$ with $t_1 \neq t_2$, $\Longrightarrow \gamma(t_1) \neq \gamma(t_2)$

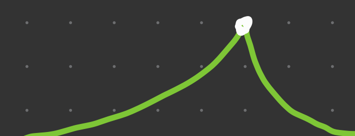
- $\gamma \in C^1([a,b], \mathbb{R}^n)$
- $\|\gamma'(t)\| = \left(\gamma_1'(t)^2 + \gamma_2'(t)^2 + \dots + \gamma_n'(t)^2\right)^{1/2} \neq 0 \quad \forall t \in [a,b]$





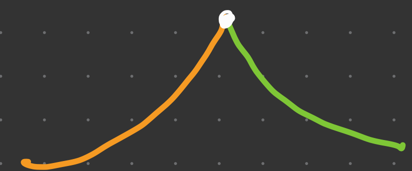






- Definition 2: $\Gamma \subset \mathbb{R}^n$ is a <u>closed regular simple curve</u> if it is a regular simple curve and $\gamma(a) = \gamma(b)$

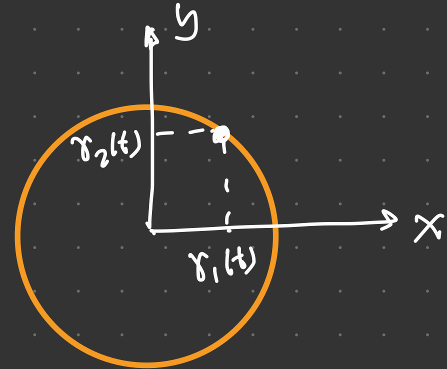
- Definition 3: $\Gamma \subset \mathbb{R}^n$ is <u>piecewise regular simple curve</u> if there exist $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ that are regular simple curves such that $\Gamma = \bigcup_{i=1}^{k} \Gamma_i$



2.1.2 Examples:

- Example 1: $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$$t \longmapsto \gamma(t) = (\cos t, \sin t)$$



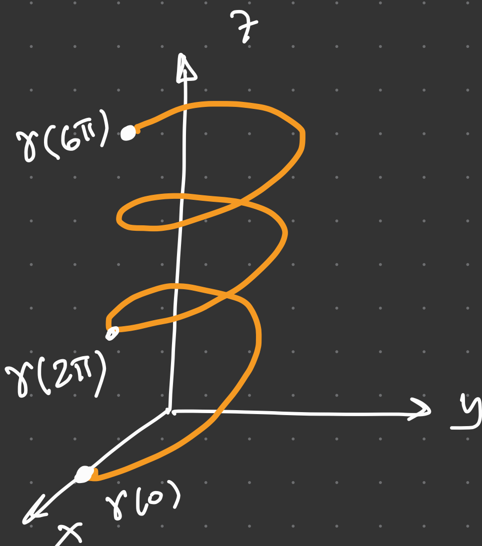
Γ: circle in $\mathbb{R}^2$, radius $R=1$, and center $\binom{0}{0}$

$$\gamma'(t) = (-\sin t, \cos t)$$

$$\|\gamma'(t)\| = \left((-\sin t)^2 + (\cos t)^2\right)^{1/2} = 1 \neq 0 \ \forall t.$$

- Example 2:

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^3$$
$$t \longmapsto \gamma(t) = (3\cos t, 3\sin t, 4t)$$



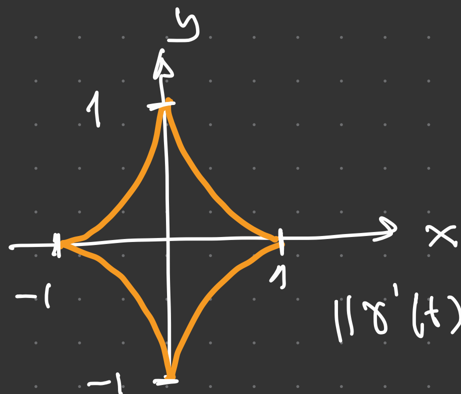
Γ: circular helix in $\mathbb{R}^3$, radius 3,

$$\gamma'(t) = (-3\sin t, 3\cos t, 4)$$

$$\|\gamma'(t)\| = 5 \quad \forall t$$

• Example 3: $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$t \longmapsto \gamma(t) = (\cos^3 t, \sin^3 t)$



Γ: astroide in $\mathbb{R}^2$

$$\gamma'(t) = (-3\cos^2 t \sin t, \; 3\sin^2 t \cos t)$$

$$\|\gamma'(t)\| = (9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t)^{1/2}$$

$$= 3\,|\cos t \sin t|$$

$\|\gamma'(t)\| = 0$ for $t = 0, t = \pi/2, t = 3\pi/2, 2\pi \ldots \longrightarrow$ the curve is <u>non-regular</u>

2.2 Curvilinear integrals

2.2.1 Definition

Let $\Gamma \subset \mathbb{R}^n$ be a regular simple curve with a parameterization

$$\gamma : [a,b] \longrightarrow \mathbb{R}^n$$
$$t \longmapsto \gamma(t)$$

- Definition 1 : Let $f : \Gamma \to \mathbb{R}$, $x \mapsto f(x)$ be a continuous scalar field.

The integral of f along Γ is defined as

$$\int_\Gamma f \, dl = \int_a^b f(\gamma(t)) \, \|\gamma'(t)\| \, dt$$

Remark: The curve's length can be computed by setting $f \equiv 1$.

$$\text{length}(\Gamma) = \int_\Gamma 1 \, dl = \int_\Gamma dl = \int_a^b \underbrace{f(\gamma(t))}_{1} \|\gamma'(t)\| \, dt$$

$$= \int_a^b \|\gamma'(t)\| \, dt$$

- Definition 2: let $F : \Gamma \to \mathbb{R}^n$

$$x \mapsto F(x) = (F_1(x), \dots, F_n(x))$$

be a continuous vector field. The <u>integral of F along Γ</u> is

$$\int_\Gamma F \cdot dl = \int_a^b F(\gamma(t)) \cdot \gamma'(t) \, dt = \int_a^b \sum_{i=1}^n F_i(\gamma(t)) \cdot \gamma_i'(t) \, dt$$

- Definition 3: If Γ is piecewise regular simple, $\Gamma = \bigcup_{i=1}^k \Gamma_i$

$$\int_\Gamma f \, dl = \sum_{i=1}^k \int_{\Gamma_i} f \, dl \quad \text{and} \quad \int_\Gamma F \cdot dl = \sum_{i=1}^k \int_{\Gamma_i} F \cdot dl$$

2.2.2 Examples

- Example 1: Compute $\int_\Gamma f \, dl$ for

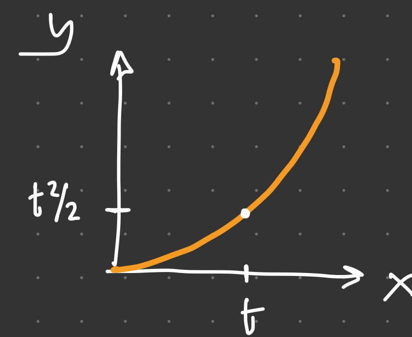
a) $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto f(x,y) = \sqrt{x^2 + 4y^2}$

and the curve Γ is parameterized as:

$\gamma: [0, 1] \longrightarrow \mathbb{R}^2$

$t \longmapsto \gamma(t) = \left(t, \frac{t^2}{2}\right)$



$$\int_\Gamma f \, dl = \int_0^1 f(\gamma(t)) \, \|\gamma'(t)\| \, dt$$

$\gamma'(t) = (1, t)$, $\|\gamma'(t)\| = \sqrt{1 + t^2}$

$$f(\gamma(t)) = \sqrt{t^2 + 4\left(\frac{t^2}{2}\right)^2}$$

$$\int_\Gamma f\,d\ell = \int_0^1 \sqrt{t^2+4\left(\frac{t^2}{2}\right)^2}\ \sqrt{1+t^2}\,dt = \int_0^1 t\sqrt{1+t^2}\sqrt{1+t^2}\,dt$$

$$= \int_0^1 (t+t^3)\,dt = \frac{t^2}{2}\Big|_0^1 + \frac{t^4}{4}\Big|_0^1 = \frac{3}{4}.$$

b) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto f(x,y) = x$

and the curve $\Gamma = \{(x,y) \in \mathbb{R}^2 : y = \cosh x \text{ for } x \in [0,1]\}$

Parameterization of Γ: $\gamma : [0,1] \longrightarrow \mathbb{R}^2$

$t \longmapsto \gamma(t) = (t, \cosh t)$

$\gamma'(t) = (1, \sinh t)$, $\|\gamma'(t)\| = \sqrt{1+\sinh^2 t} = \cosh t$

$f(\gamma(t)) = t.$

$u = t, \; v = \sinh t$

$$\int_\Gamma f \, dl = \int_0^1 t \cosh t \, dt = t \sinh t \Big|_0^1 - \int_0^1 \sinh t \, dt$$

$$= \sinh(1) - \cosh t \Big|_0^1 = \sinh(1) - \cosh(1) + 1 =$$

$$\frac{e - e^{-1}}{2} + \frac{e + e^{-1}}{2} + 1 = \frac{e-1}{e}$$

- Example 2:

a) $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x, y, z) \longmapsto F(x, y, z) = (x, z, y)$

$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^3$

$t \longmapsto \gamma(t) = (\cos t, \sin t, 0)$

$$\gamma'(t) = (-\sin t, \cos t, 0)$$

$$\int_\Gamma F \cdot dl = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$F(\gamma(t)) = (\cos t, 0, \sin t)$$

$$\int_\Gamma F \cdot dl = \int_0^{2\pi} (\cos t, 0, \sin t) \cdot (-\sin t, \cos t, 0)\, dt = \int_0^{2\pi} -\cos t \sin t\, dt$$

1) $$= -\int_0^{2\pi} \cos t \sin t\, dt = -\frac{1}{2} \sin^2 t \Big|_0^{2\pi} = 0$$

2) $$= -\frac{1}{2} \int_0^{2\pi} \sin(2t)\, dt = \frac{1}{4} \cos(2t) \Big|_0^{2\pi} = 0.$$