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Proof for $n=3$

1) $\operatorname{grad} f : \Omega \to \mathbb{R}^3$ $\quad G_i = \frac{\partial f}{\partial x_i}$, $i = 1, 2, 3$

$$\operatorname{div}(\operatorname{grad} f) = \operatorname{div}\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) = \operatorname{div}(G_1, G_2, G_3)$$

$$= \frac{\partial G_1}{\partial x_1} + \frac{\partial G_2}{\partial x_2} + \frac{\partial G_3}{\partial x_3} = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} = \Delta f \quad \square$$

3)

$$\operatorname{curl}(\operatorname{grad} f) = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

4) $\operatorname{curl} F : \Omega \to \mathbb{R}^3$

$$\operatorname{div}(\operatorname{curl} F) = \operatorname{div}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 F_2}{\partial x \partial z}} + \cancel{\frac{\partial^2 F_1}{\partial y \partial z}} - \cancel{\frac{\partial^2 F_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 F_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 F_1}{\partial z \partial y}} = 0. \quad \square$$

- $f \longrightarrow F = \operatorname{grad} f$, $\quad \operatorname{curl} F = \operatorname{curl}(\operatorname{grad} f) = 0.$
- $\underline{\Delta f} = \operatorname{div}(\operatorname{grad} f) = \underline{\operatorname{div}(F)} = 0$