EPFL - Fall 2024	Dr. Pablo Antolin
Analysis III - 202 (c) GM EL MX	**Exercises**
Series 12	December 3, 2024

Note: several exercises are extracted from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Hint: For exercises 1, 2, and 3, look for a solution given as a Fourier series. Using the initial equation, find the equations containing the Fourier coefficients as the unknowns. Solve these equations to get the Fourier coefficients.

Exercise 1 (Ex 17.10 page 272)**.**
Find a 2π-periodic solution $y(x)$ for the following differential equation:

$$y'(x) + y(x) = f(x)$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function defined by

$$f(x) = \begin{cases} \left(x - \frac{\pi}{2}\right)^2 & \text{if } 0 \leq x < \pi, \\ \dfrac{\pi^2}{2} - \left(x - \frac{3\pi}{2}\right)^2 & \text{if } \pi \leq x < 2\pi. \end{cases}$$

Exercise 2 (Ex 17.8 page 271)**.**

1. Calculate the Fourier series for the even and 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \sin(3x) \quad \text{if} \quad x \in [0, \pi].$$

2. Find a function y defined by $y(x)$, even and 2π-periodic that verifies the equation:
$$y(x) - 2y(x - \pi) = \sin(3x) \quad \text{for} \quad x \in [0, \pi].$$

Exercise 3 (Exemple 17.6 page 268)**.**
Find a 2π-periodic solution $y(x)$ for the following equation:

$$y(x) + 2y(x - \pi) = \cos x + 3\sin(2x) + 4\cos(5x) \qquad \text{for} \qquad 0 \leq x \leq 2\pi.$$

Hint: For all $n \in \mathbb{N}$, we have the trigonometric formulas

$$\cos(nt - n\pi) = (-1)^n \cos(nt), \quad \sin(nt - n\pi) = (-1)^n \sin(nt).$$

Exercise 4 (Ex 15.1 page 239)**.**
Compute the Fourier transform for the function f defined by:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-x} & \text{if } x > 0. \end{cases}$$

Exercise 5.
Compute the Fourier transform of the function f defined by:

$$f : \mathbb{R} \to \mathbb{R};\ x \mapsto f(x) = \begin{cases} 1 & \text{if } x \in [-1, 1[, \\ 0 & \text{else.} \end{cases}$$

Exercise 6.
Draw the graph for the following functions and find their Fourier transforms:

a) $f(x) = \begin{cases} \sqrt{\frac{\pi}{2}}, & \text{if } -1 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$

b) $g(x) = \begin{cases} \pi + \frac{\pi}{2}x, & \text{if } -2 \leq x \leq 0, \\ \pi - \frac{\pi}{2}x, & \text{if } 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$

Which is the link between the functions f and g ?

Exercise 7 (Ex 15.2 page 239)**.**
We consider the function $f\colon \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) = e^{-x}\cos x$.

a) Compute the Fourier cosine transform of the even extension to $\mathbb{R}$ of f.

b) Compute the Fourier sine transform of the odd extension to $\mathbb{R}$ of f.