



Teacher : Pablo Antolin
Analysis III - Mock exam - Student
November 2024
Duration : 180 minutes

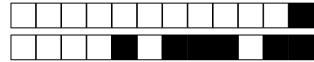
Student One

SCIPER: **111111**

Do not turn the page before the start of the exam. This document is double-sided, has 30 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
 - +2 points if your answer is correct,
 - 0 points if your answer is incorrect, you give no answer, or more than one answer is marked.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

| Respectez les consignes suivantes Observe this guidelines Beachten Sie bitte die unten stehenden Richtlinien | | |
|---|--|---|
| choisir une réponse select an answer Antwort auswählen | ne PAS choisir une réponse NOT select an answer NICHT Antwort auswählen | Corriger une réponse Correct an answer Antwort korrigieren |
|       | | |
| ce qu'il ne faut PAS faire what should NOT be done was man NICHT tun sollte | | |
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First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 The non-zero complex Fourier coefficients of the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$g(x) = \cos(x) + 3 \sin(3x),$$

which enables to express g as:

$$g(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

are

- c_1, c_{-1}, c_3, c_{-3}
- c_1, c_{-3}
- c_1, c_3
- $c_0, c_1, c_{-1}, c_3, c_{-3}$

Question 2 Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the vector field defined by:

$$F(x, y, z) = (x^2 + y^2 + z^2, xy, z).$$

Then:

- $\operatorname{div}(\operatorname{curl}(F)) = 0$ over \mathbb{R}^3
- $\operatorname{div}(F) = 0$ over \mathbb{R}^3
- $F = \nabla f$, for any $f \in C^1(\mathbb{R}^3)$
- $\operatorname{curl}(F) = 0$ over \mathbb{R}^3

Question 3 Let F be the vector field defined by:

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2; (x, y) \mapsto (x, y),$$

and let $R \in \mathbb{R}, R > 0$ and A be the domain defined by:

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\}.$$

We also denote the boundary of A by ∂A , and the outer unit normal of ∂A by $\nu : \partial A \rightarrow \mathbb{R}^2$.

The integral $\int_{\partial A} F \cdot \nu \, dl$ is equal to:

- 0
- $2\pi R^2$
- $\frac{3}{4}\pi R$
- $\frac{3}{5}\pi^2 R$



Question 4 Let F be the vector field defined by:

$$F : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2; (x, y) \mapsto \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

F is conservative (i.e. it derives from a potential)

- over $\Omega = \{(x, y) : 1 \leq x \leq 3, 2 \leq y \leq 10\}$.
- over $\Omega = \{(x, y) : 2 \leq x^2 + y^2 \leq 4\}$.
- over $\Omega = \{(x, y) : x^2 + y^2 \leq 10\}$.
- for any domain Ω .

Question 5 Let $T > 0$, and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by:

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{T}{2}[\\ -1 & \text{if } x \in [\frac{T}{2}, T[\end{cases}$$

extended by T -periodicity to \mathbb{R} . Its Fourier series is:

$$Ff(x) = \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)} \sin\left(\frac{2\pi}{T}(2n+1)x\right)$$

The sum $\sum_{n=0}^{\infty} \frac{1}{n^2 + n + 1/4}$ is equal to:

- 0
- $\pi^2/2$
- $\pi^2/16$
- $\pi^2/8$

Question 6 Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 -periodic solution of the following system:

$$\begin{cases} u'(x) + 3u(x) = \cos(3x) + \sin(5x), & \forall x \in \mathbb{R} \\ u(0) = u(2\pi) \\ u'(0) = u'(2\pi) \end{cases}$$

Then:

- $u(x) = \frac{1}{8} \sin(3x) + \frac{1}{6} \cos(3x) - \frac{5}{34} \sin(5x) + \frac{3}{28} \cos(5x)$
- $u(x) = \frac{1}{6} \sin(3x) + \frac{1}{6} \cos(3x) + \frac{3}{34} \sin(5x) - \frac{5}{34} \cos(5x)$
- $u(x) = \frac{1}{8} \sin(3x) + \frac{1}{8} \cos(3x) + \frac{3}{34} \sin(5x) - \frac{3}{28} \cos(5x)$
- $u(x) = \frac{1}{8} \sin(3x) + \frac{3}{28} \cos(5x)$



Question 7 Let f be the scalar field defined by:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}; (x, y) \mapsto xy + x + 1,$$

and let $R \in \mathbb{R}, R > 0$, and Γ the curved defined by:

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = R^2\}.$$

The integral $\int_{\Gamma} f \, dl$ is equal to:

- 0
- $2\pi R$
- $2\pi R^2 + \pi R + 1$
- $2\pi R^3 + \pi R^2 + R$

Question 8 Consider the functions $g : \mathbb{R} \rightarrow \mathbb{R}$ and $h : \mathbb{R} \rightarrow \mathbb{R}$ defined such that:

$$g(x) = e^{-\frac{x^2}{2}}, \quad \text{and } h(x) = g\left(\frac{x}{4}\right).$$

Then, the Fourier transform $\hat{h} = \mathcal{F}(h)$ verifies:

- $\hat{h}'(\alpha) = i\frac{\alpha}{4}e^{-\frac{\alpha^2}{32}}$
- $\int_{-\infty}^{\infty} |\hat{h}'(\alpha)| d\alpha = \sqrt{2\pi}$
- $\hat{h}(\alpha) = e^{-8\alpha^2}$
- $\left(\frac{1}{2} \frac{d}{d\alpha} \left[e^{4\alpha^2} \hat{h}(\alpha)\right]\right)^2 = 64\alpha^2 \hat{h}(\alpha)$



Second, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 8: *This question is worth 9 points.*

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(i) Let Γ be the curve defined by

$$\Gamma = \left\{ \left(\frac{1}{3}t^3, 3t, \frac{\sqrt{6}t^2}{2} \right) \mid t \in [-1, 1] \right\}.$$

Compute the length of Γ .

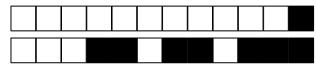
(ii) Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the vector field defined by

$$F(x, y) = (x^2, y \cos(x^2))$$

and Ω the triangle whose vertices are $(0, 0)$, $(\sqrt{\pi/2}, 0)$, and $(\sqrt{\pi/2}, \sqrt{\pi/2})$. Compute

$$\int_{\partial\Omega} F \cdot \nu dl$$

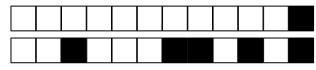
where $\nu : \partial\Omega \rightarrow \mathbb{R}^2$ is outer unit normal field of the boundary of Ω .



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Question 9: This question is worth 6 points.

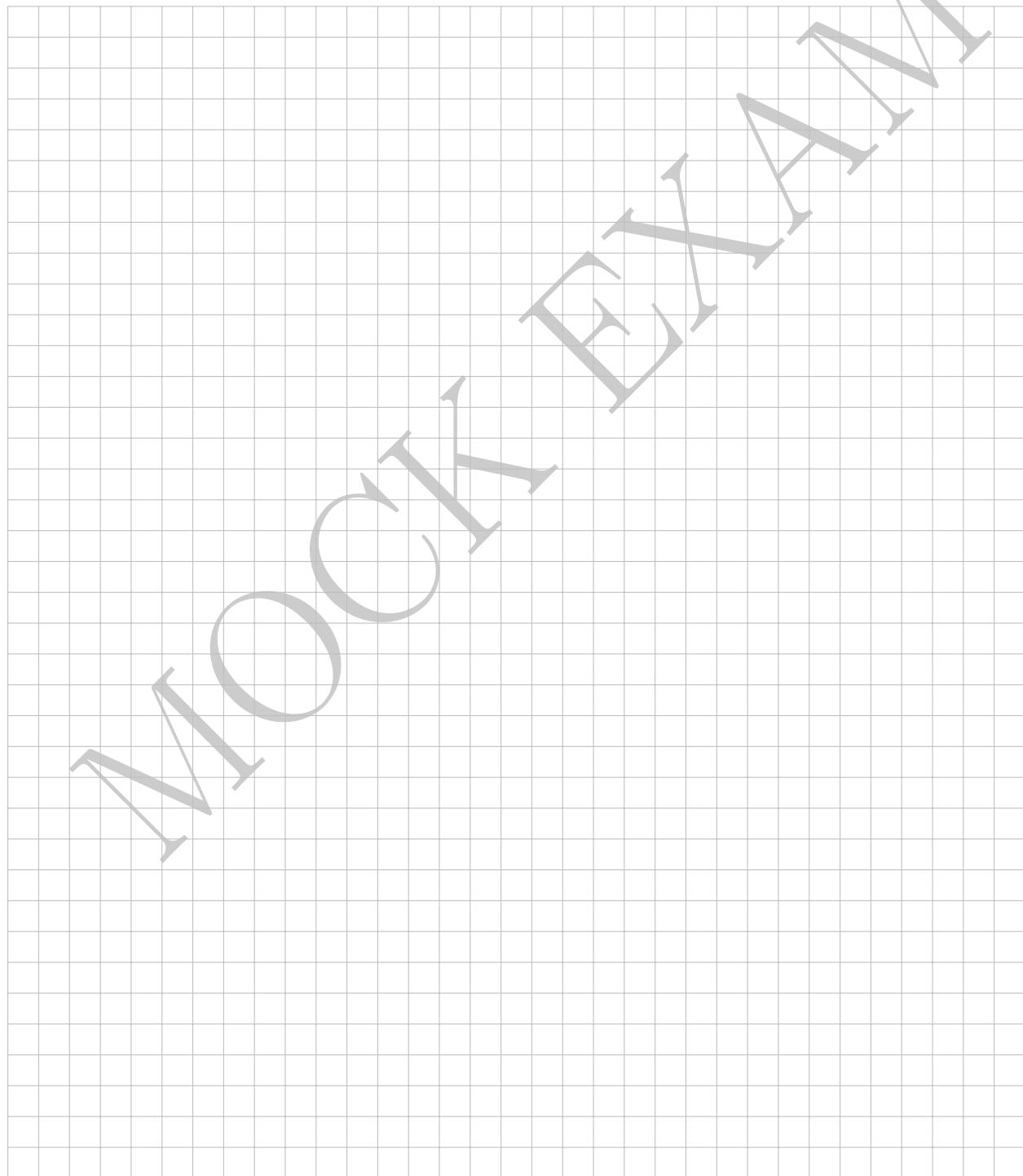
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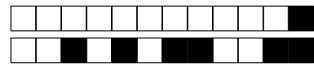
Let $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$ and $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$F(x, y) = \left(\frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2} \right).$$

(i) Compute $\operatorname{curl} F$.

(ii) Determine if F derives from a potential in Ω . If it does, find a potential of F , otherwise, justify why it does not derive from a potential in Ω .

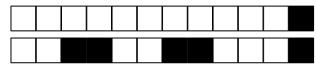




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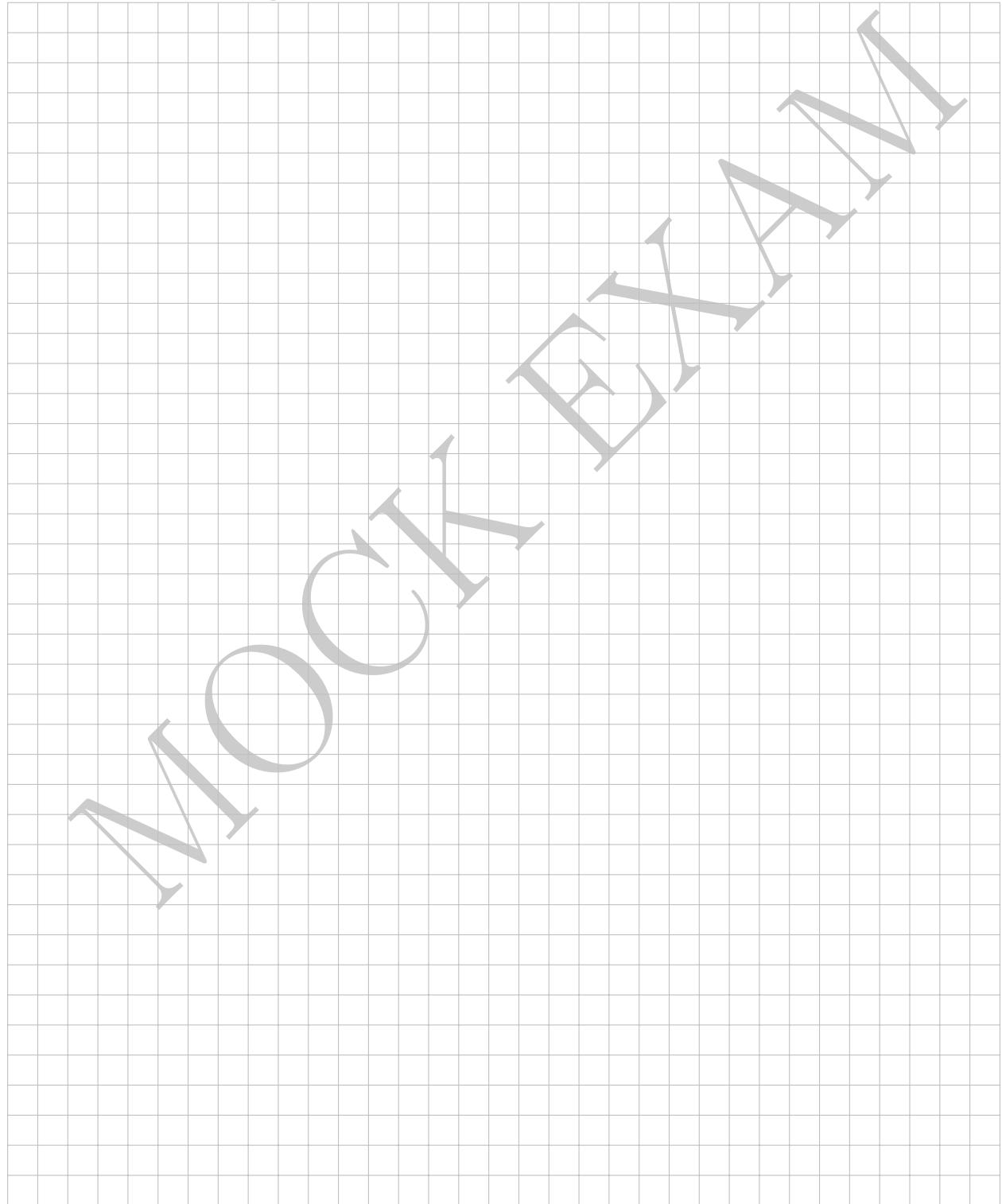
Question 10: This question is worth 3 points.

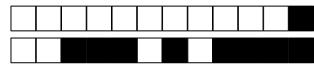
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Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$; $F(x, y) = (F_1(x, y), F_2(x, y))$, be a vector field such that $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and $\operatorname{div} F = 0$. Let $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a vector field defined by:

$$G(x, y) = (F_2(-x, y), F_1(-x, y)).$$

Show that G derives from a potential in \mathbb{R}^2 .





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Question 11: This question is worth 14 points.

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Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the vector field defined as $F(x, y, z) = (0, x, 0)$ and let Σ be the surface defined by

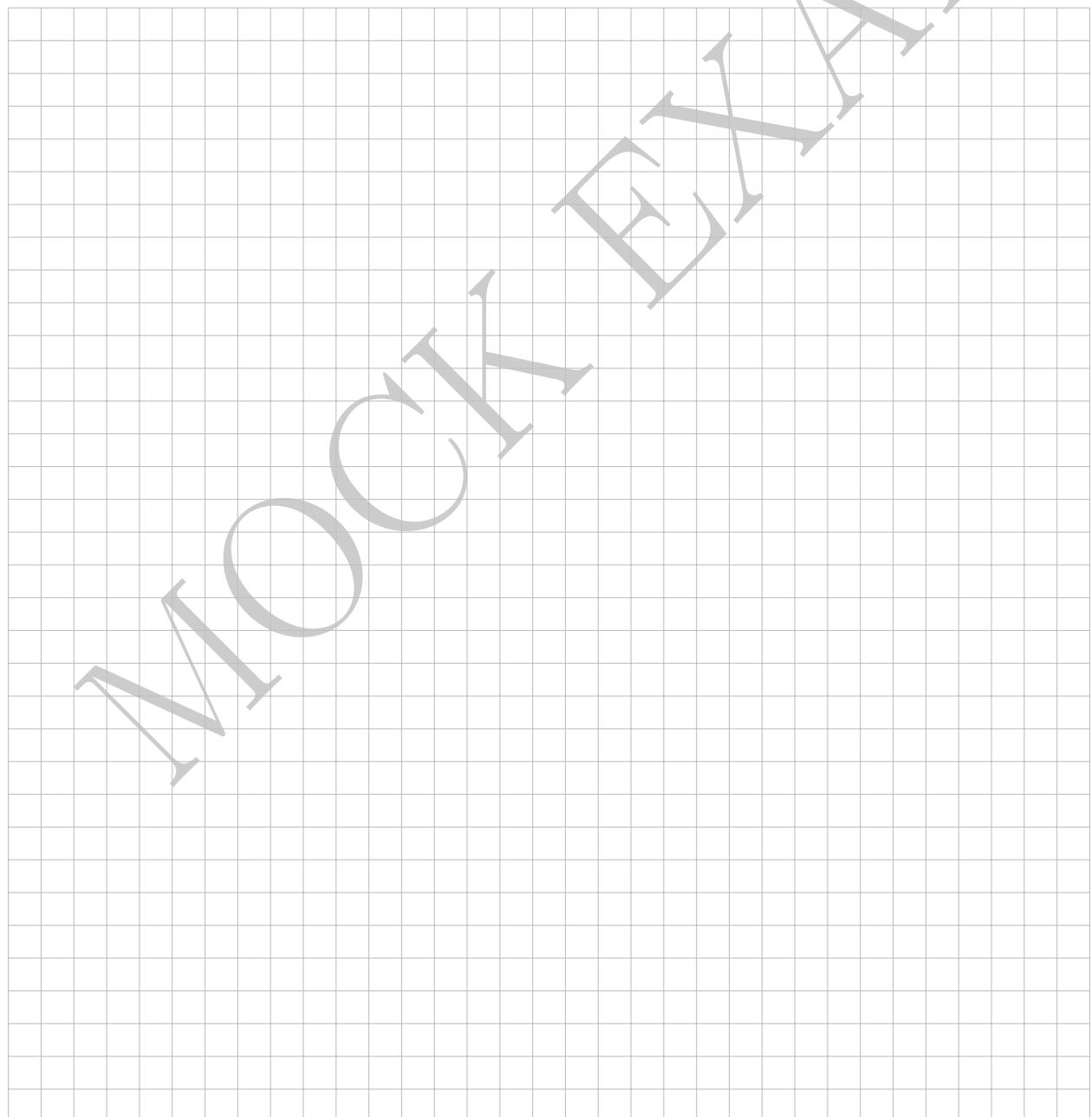
$$\Sigma = \left\{ (x, y, z) \in \mathbb{R}^3 \mid z = \left(\sqrt{x^2 + y^2} + 1 \right) \left(3 - \sqrt{x^2 + y^2} \right), y \geq 0, z \geq 0 \right\}.$$

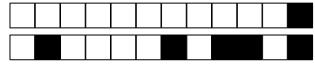
Verify the Stokes theorem for F and Σ .

Note: if necessary, use the following formulas:

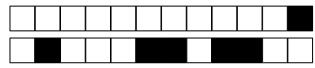
$$\cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x)$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x)$$

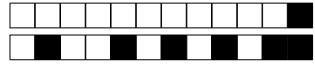




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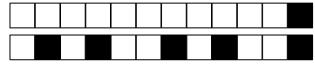
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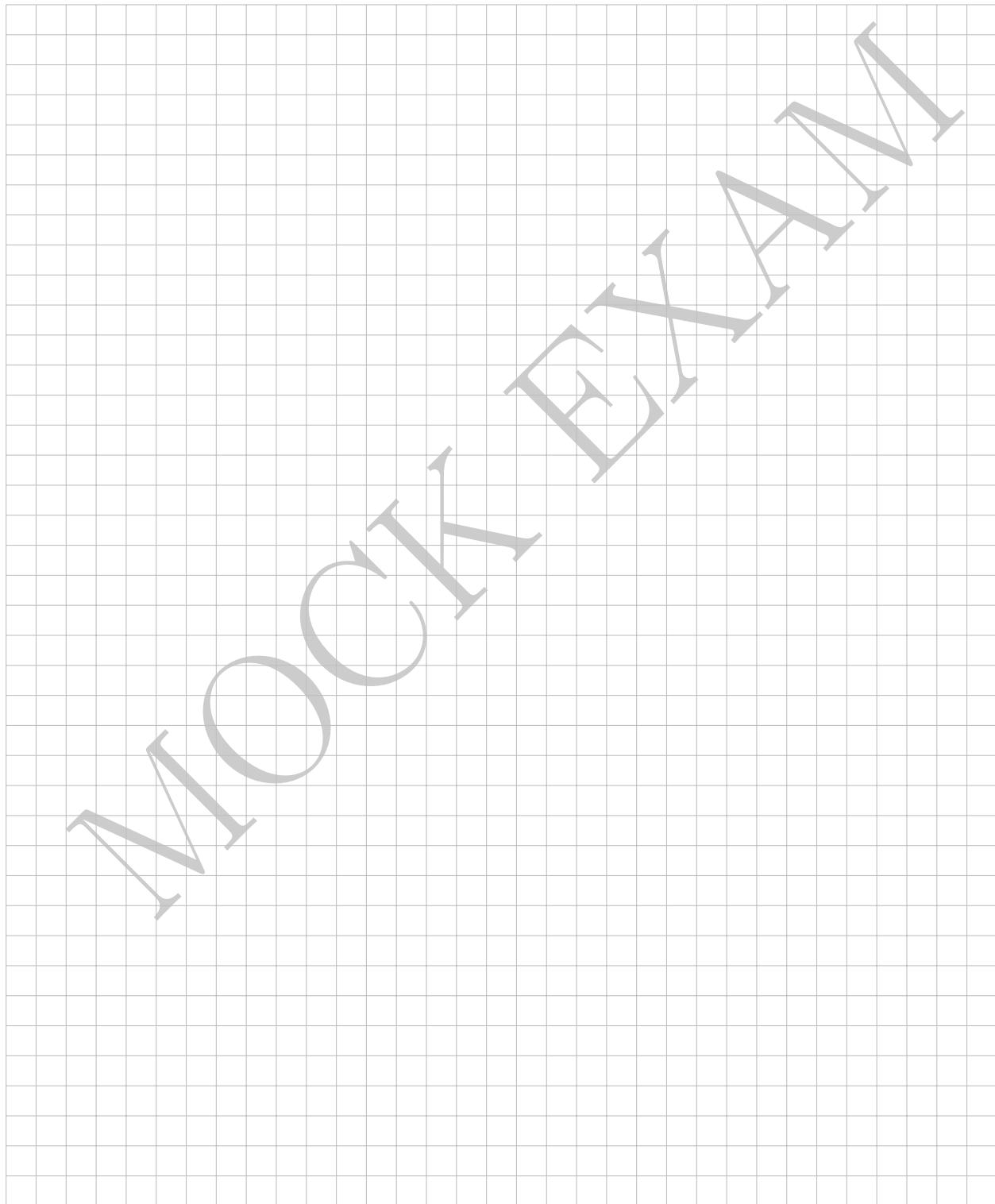


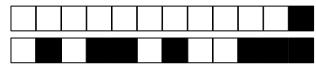
Question 12: This question is worth 9 points.

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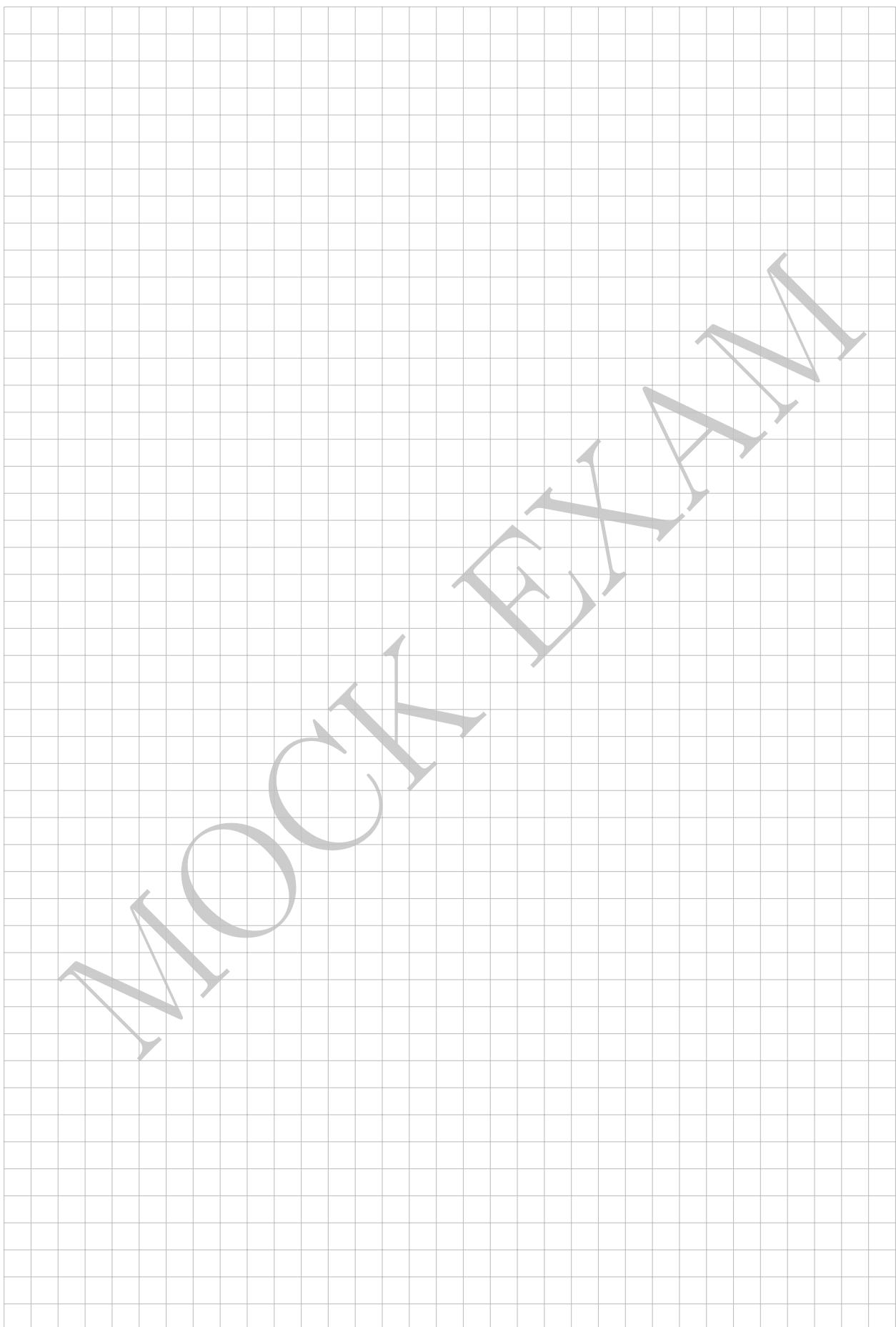
Let $f : [0, \pi] \rightarrow \mathbb{R}$ be the function $f(x) = -x^2 + 2\pi x$.

- (i) Compute $F_s f$, the Fourier series in sines of f .
- (ii) Using the course's results, compare $F_s f$ and f in the interval $[0, \pi]$.





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Question 13: This question is worth 4 points.

0 1 2 3 4

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by

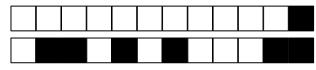
$$g(x) = \begin{cases} -x & \text{if } -\pi \leq x \leq 0 \\ \pi & \text{if } 0 < x < \pi \end{cases} \quad \text{extended by } 2\pi\text{-periodicity.}$$

The real Fourier coefficients of g are

$$a_0 = \frac{3\pi}{2};$$
$$a_n = \begin{cases} -\frac{2}{n^2\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \quad \text{for } n \geq 1;$$
$$b_n = \frac{1}{n} \quad \text{for } n \geq 1.$$

Using those coefficients and one result of the course, compute the sum

$$\sum_{k=1}^{+\infty} \frac{1}{(k - \frac{1}{2})^2}.$$



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Question 14: This question is worth 8 points.

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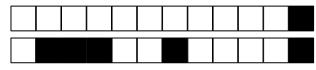
(i) Write the definition of the Fourier transform of a function detailing its hypotheses

(ii) Using the properties of the Fourier transform, find $u : \mathbb{R} \rightarrow \mathbb{R}$, the solution of

$$-10u(x) + \int_{-\infty}^{+\infty} (9u(t) - 4u''(t)) e^{-\frac{3}{2}|x-t|} dt = \frac{4x^2}{(2\pi + x^2)^2}.$$

If needed, use the Fourier transforms of the table below.

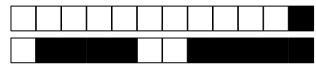
| | $f(y)$ | $\mathcal{F}(f)(\alpha) = \hat{f}(\alpha)$ |
|----|---|---|
| 1 | $f(y) = \begin{cases} 1, & \text{si } y < b \\ 0, & \text{sinon} \end{cases}$ | $\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(b \alpha)}{\alpha}$ |
| 2 | $f(y) = \begin{cases} 1, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$ | $\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$ |
| 3 | $f(y) = \begin{cases} e^{-wy}, & \text{si } y > 0 \\ 0, & \text{sinon} \end{cases} \quad (w > 0)$ | $\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{w + i\alpha}$ |
| 4 | $f(y) = \begin{cases} e^{-wy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$ | $\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(w+i\alpha)b} - e^{-(w+i\alpha)c}}{w + i\alpha}$ |
| 5 | $f(y) = \begin{cases} e^{-iwy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$ | $\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(w+\alpha)b} - e^{-i(w+\alpha)c}}{w + \alpha}$ |
| 6 | $f(y) = \frac{1}{y^2 + w^2} \quad (w \neq 0)$ | $\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{- w\alpha }}{ w }$ |
| 7 | $f(y) = \frac{e^{- wy }}{ w } \quad (w \neq 0)$ | $\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + w^2}$ |
| 8 | $f(y) = e^{-w^2 y^2} \quad (w \neq 0)$ | $\hat{f}(\alpha) = \frac{1}{\sqrt{2} w } e^{-\frac{\alpha^2}{4w^2}}$ |
| 9 | $f(y) = y e^{-w^2 y^2} \quad (w \neq 0)$ | $\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2} w ^3} e^{-\frac{\alpha^2}{4w^2}}$ |
| 10 | $f(y) = \frac{4y^2}{(y^2 + w^2)^2} \quad (w \neq 0)$ | $\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{ w } - \alpha \right) e^{- w\alpha }$ |



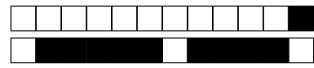
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