EPFL

Teacher : Pablo Antolin
Analysis III - Mock exam - Student
November 2024
Duration : 180 minutes

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Student One

SCIPER : **111111**

Do not turn the page before the start of the exam. This document is double-sided, has 30 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
  +2 points if your answer is correct,
  0 points if your answer is incorrect, you give no answer, or more than one answer is marked.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer Antwort auswählen

ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen

Corriger une réponse | Correct an answer Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 The non-zero complex Fourier coefficients of the function $g : \mathbb{R} \to \mathbb{R}$ defined by:

$$g(x) = \cos(x) + 3\sin(3x),$$

which enables to express g as:

$$g(x) = \sum_{k=-\infty}^{\infty} = c_k e^{ikx}$$

are

- ☐ c_1, c_{-1}, c_3, c_{-3}
- ☐ c_1, c_{-3}
- ☐ c_1, c_3
- ☐ c_0, c_1, c_{-1}, c_3, c_{-3}

Question 2 Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by:

$$F(x, y, z) = (x^2 + y^2 + z^2, xy, z).$$

Then:

- ☐ $\operatorname{div}(\operatorname{curl}(F)) = 0$ over $\mathbb{R}^3$
- ☐ $\operatorname{div}(F) = 0$ over $\mathbb{R}^3$
- ☐ $F = \nabla f$, for any $f \in C^1(\mathbb{R}^3)$
- ☐ $\operatorname{curl}(F) = 0$ over $\mathbb{R}^3$

Question 3 Let F be the vector field defined by:

$$F : \mathbb{R}^2 \to \mathbb{R}^2;\ (x, y) \mapsto (x, y),$$

and let $R \in \mathbb{R}, R > 0$ and A be the domain defined by:

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\right\}.$$

We also denote the boundary of A by ∂A, and the outer unit normal of ∂A by $\nu : \partial A \to \mathbb{R}^2$. The integral $\int_{\partial A} F \cdot \nu \, dl$ is equal to:

- ☐ 0
- ☐ $2\pi R^2$
- ☐ $\frac{3}{4}\pi R$
- ☐ $\frac{3}{5}\pi^2 R$

Question 4 Let F be the vector field defined by:

$$F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2;\ (x, y) \mapsto \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

F is conservative (*i.e.* it derives from a potential)

- ☐ over $\Omega = \{(x, y) :\ 1 \le x \le 3, 2 \le y \le 10\}$.
- ☐ over $\Omega = \{(x, y) :\ 2 \le x^2 + y^2 \le 4\}$.
- ☐ over $\Omega = \{(x, y) :\ x^2 + y^2 \le 10\}$.
- ☐ for any domain Ω.

Question 5 Let $T > 0$, and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \left[0, \frac{T}{2}\right[\\ -1 & \text{if } x \in \left[\frac{T}{2}, T\right[\end{cases}$$

extended by T-periodicity to $\mathbb{R}$. Its Fourier series is:

$$Ff(x) = \sum_{n=0}^{\infty} \frac{4}{\pi\,(2n+1)} \sin\left(\frac{2\pi}{T}(2n+1)x \right)$$

The sum $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2 + n + 1/4}$ is equal to:

- ☐ 0
- ☐ $\pi^2/2$
- ☐ $\pi^2/16$
- ☐ $\pi^2/8$

Question 6 Let $u : \mathbb{R} \to \mathbb{R}$ be a C^1-periodic solution of the following system:

$$\begin{cases} u'(x) + 3u(x) = \cos(3x) + \sin(5x), \quad \forall x \in \mathbb{R} \\ u(0) = u(2\pi) \\ u'(0) = u'(2\pi) \end{cases}$$

Then:

- ☐ $u(x) = \frac{1}{8}\sin(3x) + \frac{1}{6}\cos(3x) - \frac{5}{34}\sin(5x) + \frac{3}{28}\cos(5x)$
- ☐ $u(x) = \frac{1}{6}\sin(3x) + \frac{1}{6}\cos(3x) + \frac{3}{34}\sin(5x) - \frac{5}{34}\cos(5x)$
- ☐ $u(x) = \frac{1}{8}\sin(3x) + \frac{1}{8}\cos(3x) + \frac{3}{34}\sin(5x) - \frac{3}{28}\cos(5x)$
- ☐ $u(x) = \frac{1}{8}\sin(3x) + \frac{3}{28}\cos(5x)$

Question 7 Let f be the scalar field defined by:

$$f : \mathbb{R}^2 \to \mathbb{R};\ (x, y) \mapsto xy + x + 1,$$

and let $R \in \mathbb{R}, R > 0$, and Γ the curved defined by:

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = R^2\right\}.$$

The integral $\int_\Gamma f \,\mathrm{d}l$ is equal to:

- ☐ 0
- ☐ $2\pi R$
- ☐ $2\pi R^2 + \pi R + 1$
- ☐ $2\pi R^3 + \pi R^2 + R$

Question 8 Consider the functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ defined such that:

$$g(x) = e^{-\frac{x^2}{2}}, \quad \text{and } h(x) = g\left(\frac{x}{4}\right).$$

Then, the Fourier transform $\hat{h} = \mathcal{F}(h)$ verifies:

- ☐ $\hat{h}'(\alpha) = i\frac{\alpha}{4}e^{-\frac{\alpha^2}{32}}$
- ☐ $\int_{-\infty}^{\infty} |\hat{h}'(\alpha)| \mathrm{d}\alpha = \sqrt{2\pi}$
- ☐ $\hat{h}(\alpha) = e^{-8\alpha^2}$
- ☐ $\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[e^{4\alpha^2}\hat{h}(\alpha)\right]\right)^2 = 64\alpha^2\hat{h}(\alpha)$

Second, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 8: *This question is worth 9 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9

(i) Let Γ be the curve defined by

$$\Gamma = \left\{ \left(\frac{1}{3}t^3, 3t, \frac{\sqrt{6}t^2}{2} \right) \,\middle|\, t \in [-1, 1] \right\}.$$

Compute the length of Γ.

(ii) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$F(x, y) = \big(x^2, y\cos(x^2)\big)$$

and Ω the triangle whose vertices are $(0, 0)$, $(\sqrt{\pi/2}, 0)$, and $(\sqrt{\pi/2}, \sqrt{\pi/2})$. Compute

$$\int_{\partial\Omega} F \cdot \nu \mathrm{d}l$$

where $\nu : \partial\Omega \to \mathbb{R}^2$ is outer unit normal field of the boundary of Ω.

Question 9: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6

Let $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x,y) = \left(\frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2} \right).$$

(i) Compute curl F.

(ii) Determine if F derives from a potential in Ω. If it does, find a potential of F, otherwise, justify why it does not derive from a potential in Ω.

Question 10: *This question is worth 3 points.*

☐0 ☐1 ☐2 ☐3

Let $F : \mathbb{R}^2 \to \mathbb{R}^2; F(x, y) = (F_1(x, y), F_2(x, y))$, be a vector field such that $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and div $F = 0$. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field defined by:

$$G(x, y) = (F_2(-x, y), F_1(-x, y)) .$$

Show that G derives from a potential in $\mathbb{R}^2$.

Question 11: *This question is worth 14 points.*

0 1 2 3 4 5 6 7
8 9 10 11 12 13 14

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined as $F(x, y, z) = (0, x, 0)$ and let Σ be the surface defined by

$$\Sigma = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; z = \left(\sqrt{x^2 + y^2} + 1\right)\left(3 - \sqrt{x^2 + y^2}\right), y \geq 0, z \geq 0 \right\}.$$

Verify the Stokes theorem for F and Σ.
Note: if necessary, use the following formulas:

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

Question 12: *This question is worth 9 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9

Let $f : [0, \pi] \to \mathbb{R}$ be the function $f(x) = -x^2 + 2\pi x$.

(i) Compute $F_s f$, the Fourier series in sines of f.

(ii) Using the course's results, compare $F_s f$ and f in the interval $[0, \pi]$.

Question 13: *This question is worth 4 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

Let $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} -x & \text{if} \quad -\pi \leq x \leq 0 \\ \pi & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{extended by } 2\pi\text{-periodicity.}$$

The real Fourier coefficients of g are

$$a_0 = \frac{3\pi}{2};$$

$$a_n = \begin{cases} -\frac{2}{n^2\pi} & \text{if} \quad n \text{ is odd} \\ 0 & \text{if} \quad n \text{ is even} \end{cases} \qquad \text{for } n \geq 1;$$

$$b_n = \frac{1}{n} \qquad \text{for } n \geq 1.$$

Using those coefficients and one result of the course, compute the sum

$$\sum_{k=1}^{+\infty} \frac{1}{\left(k - \frac{1}{2}\right)^2}.$$

Question 14: *This question is worth 8 points.*

□ 0 □ 1 □ 2 □ 3 □ 4 □ 5 □ 6 □ 7 □ 8

(i) Write the definition of the Fourier transform of a function detailing its hypotheses

(ii) Using the properties of the Fourier transform, find $u : \mathbb{R} \to \mathbb{R}$, the solution of

$$-10u(x) + \int_{-\infty}^{+\infty} \Big(9u(t) - 4u''(t)\Big) e^{-\frac{3}{2}|x-t|} \mathrm{d}t = \frac{4x^2}{\left(2\pi + x^2\right)^2}.$$

If needed, use the Fourier transforms of the table below.

	$f(y)$	$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha)$
1	$f(y) = \begin{cases} 1, & \text{si } \lvert y\rvert < \lvert b\rvert \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lvert b\rvert \alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$
3	$f(y) = \begin{cases} e^{-wy}, & \text{si } y > 0 \\ 0, & \text{sinon} \end{cases} \quad (w > 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{w + i\alpha}$
4	$f(y) = \begin{cases} e^{-wy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(w+i\alpha)b} - e^{-(w+i\alpha)c}}{w + i\alpha}$
5	$f(y) = \begin{cases} e^{-iwy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(w+\alpha)b} - e^{-i(w+\alpha)c}}{w + \alpha}$
6	$f(y) = \frac{1}{y^2 + w^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{-\lvert w\alpha\rvert}}{\lvert w\rvert}$
7	$f(y) = \frac{e^{-\lvert wy\rvert}}{\lvert w\rvert} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + w^2}$
8	$f(y) = e^{-w^2 y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert w\rvert} e^{-\frac{\alpha^2}{4w^2}}$
9	$f(y) = y e^{-w^2 y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2}\lvert w\rvert^3} e^{-\frac{\alpha^2}{4w^2}}$
10	$f(y) = \frac{4y^2}{(y^2 + w^2)^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{\lvert w\rvert} - \lvert\alpha\rvert\right) e^{-\lvert w\alpha\rvert}$