EPFL - Fall 2024	Dr. Pablo Antolin
Analysis III - 202 (c) GM EL MX	**Correction**
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Exercise 2.
Applying the Fourier transform inversion theorem for the function f defined by $f(x) = xe^{-\omega|x|}$ (i.e., it is continuous) we obtain:

$$\begin{aligned} xe^{-\omega|x|} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathfrak{F}(f)(\alpha) e^{i\alpha x} \mathrm{d}\alpha = \frac{4\omega}{2\pi i} \int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2+\omega^2)^2} e^{i\alpha x} \mathrm{d}\alpha \\ &= \frac{2\omega}{\pi i} \Big[\int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2+\omega^2)^2} \cos(\alpha x) \mathrm{d}\alpha + i \int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2+\omega^2)^2} \sin(\alpha x) \mathrm{d}\alpha \Big]. \end{aligned}$$

As $f(x) = xe^{-\omega|x|} \in \mathbb{R}$, we get:

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2+\omega^2)^2} \cos(\alpha x) \mathrm{d}\alpha &= 0, \\ \int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2+\omega^2)^2} \sin(\alpha x) \mathrm{d}\alpha &= \frac{\pi}{2\omega} xe^{-\omega|x|}. \end{aligned}$$

The function g defined by $g(\alpha) = \dfrac{\alpha}{(\alpha^2+\omega^2)^2} \sin(\alpha x)$ is even, then we obtain:

$$\int_{0}^{+\infty} \frac{\alpha}{(\alpha^2+\omega^2)^2} \sin(\alpha x) \mathrm{d}\alpha = \frac{\pi}{4\omega} xe^{-\omega|x|}.$$

Evaluating it at $\alpha = t$ and chosing $\omega = 2$ and $x = \dfrac{1}{2}$ we obtain:

$$\int_{0}^{+\infty} \frac{t}{(t^2+4)^2} \sin\left(\frac{t}{2}\right) \mathrm{d}t = \frac{\pi}{16e}.$$