EPFL - Fall 2024	Dr. Pablo Antolin
Analysis III - 202 (c) GM EL MX	**Correction**
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Exercise 5.
The function f is piecewise-defined and we can verify that:

$$\int_{-\infty}^{+\infty} |f(x)|\, dx = \int_{-1}^{1} 1\, dx = 2 < +\infty.$$

Thus, the Fourier transform is well-defined. We compute:

$$\begin{aligned}\hat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x}\, dx = \int_{-1}^{1} e^{-i\alpha x}\, dx \\ &= -\frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\alpha x}}{i\alpha} \right]_{-1}^{1} = \frac{1}{\sqrt{2\pi}} \frac{e^{i\alpha} - e^{-i\alpha}}{i\alpha} = \sqrt{\frac{2}{\pi}} \frac{\sin\alpha}{\alpha}.\end{aligned}$$

Exercise 6.

a) The function f is piecewise-defined and integrable over $\mathbb{R}$. Its Fourier transform is

$$\begin{aligned}\hat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-i\alpha x}dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\sqrt{\frac{\pi}{2}}e^{-i\alpha x}dx \\ &= \frac{1}{2}\left[\frac{1}{-i\alpha}e^{-i\alpha x}\right]_{-1}^{1} = \frac{1}{2i\alpha}\left(e^{i\alpha}-e^{-i\alpha}\right) = \frac{\sin(\alpha)}{\alpha}.\end{aligned}$$

b) The function g is continuous and integrable over $\mathbb{R}$. Its Fourier transform is

$$\begin{aligned}\hat{g}(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(x)e^{-i\alpha x}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-2}^{0}(\pi+\frac{\pi}{2}x)e^{-i\alpha x}dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{2}(\pi-\frac{\pi}{2}x)e^{-i\alpha x}dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{-i\alpha}e^{-i\alpha x}(\pi+\frac{\pi}{2}x)\right]_{-2}^{0} - \frac{1}{\sqrt{2\pi}}\int_{-2}^{0}\frac{\pi}{-2i\alpha}e^{-i\alpha x}dx \\ &\quad + \frac{1}{\sqrt{2\pi}}\left[\frac{1}{-i\alpha}e^{-i\alpha x}(\pi-\frac{\pi}{2}x)\right]_{0}^{2} - \frac{1}{\sqrt{2\pi}}\int_{0}^{2}\frac{\pi}{2i\alpha}e^{-i\alpha x}dx \\ &= \frac{1}{\sqrt{2\pi}}\frac{\pi}{2}\left(\left[\frac{e^{-i\alpha x}}{-\alpha^2}\right]_{0}^{2} - \left[\frac{e^{-i\alpha x}}{-\alpha^2}\right]_{-2}^{0}\right) \\ &= \frac{\sqrt{\pi}}{2\sqrt{2}}\left(\frac{e^{2i\alpha}+e^{-2i\alpha}}{2}\frac{2}{-\alpha^2} - \frac{2}{-\alpha^2}\right) \\ &= \frac{\sqrt{\pi}}{-\alpha^2\sqrt{2}}\left(\cos(2\alpha)-1\right) = \sqrt{2\pi}\,\frac{\sin^2(\alpha)}{\alpha^2}.\end{aligned}$$

We observe that $\hat{g}(\alpha) = \sqrt{2\pi}\hat{f}(\alpha)^2$. So $g = f * f$.