EPFL - Fall 2024	Dr. Pablo Antolin
Analysis III - 202 (c) GM EL MX	**Correction**
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Exercise 4.

1. On one hand we have, $a_0 = \frac{2}{\pi}\int_0^\pi x\,\mathrm{d}x = \pi$ and on the other, for $n \in \mathbb{N}^*$, integrating by parts we obtain,

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\left(\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\Bigg|_{x=0}^{x=\pi}\right) = \frac{2\left((-1)^n - 1\right)}{\pi n^2};$$
$$b_n = \frac{2}{\pi}\int_0^\pi x\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\left(-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\Bigg|_{x=0}^{x=\pi}\right) = -\frac{2\,(-1)^n}{n}.$$

Thus, the Fourier cosine series of f is given by

$$F_c f\,(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2}\,\cos\left((2n+1)\,x\right),$$

and the Fourier sine series of f is given by

$$F_s f\,(x) = -2\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}\,\sin(nx).$$

2. The Fourier cosine series $F_c(f)$ corresponds to the Fourier series of a function $g : \mathbb{R} \to \mathbb{R}$ that is 2π-periodic and *even*, and equal to f over $[0,\pi[$. As the function g is continuous over $[0,\pi]$, applying Dirichlet's theorem we have

$$F_c f(x) = Fg(x) = g(x) = f(x), \qquad \text{for every } x \in [0,\pi].$$

The Fourier sine serie $F_s(f)$ corresponds to the Fourier series of a function $h : \mathbb{R} \to \mathbb{R}$ that is 2π-periodic and *odd*, and equal to f over $[0,\pi[$. The function h is continuous over $[0,\pi[$ and has a discontinuity in π. Thus, applying Dirichlet's theorem

$$F_s f(x) = Fh(x) = h(x) = f(x), \quad \text{pour tout } x \in [0,\pi[, \text{ and}$$
$$F_s f(\pi) = Fh(\pi) = \frac{h(\pi - 0) + h(\pi + 0)}{2} = 0 \neq \pi = f(\pi).$$

3. After the previous point, we find

$$\frac{\pi}{2} = f\left(\frac{\pi}{2}\right) = F_s f\left(\frac{\pi}{2}\right) = 2\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1},$$

and then $\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$. In the same way, we have

$$0 = f(0) = F_c f(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}.$$

Thus, $\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

4. Applying Parseval's identity to $F_c f = Fg$, we find,

$$\frac{a_0^2}{2} + \sum_{k=1}^{+\infty} a_k^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4} = \frac{2}{2\pi} \int_{-\pi}^{\pi} |x|^2 \mathrm{d}x = \frac{2\pi^2}{3}.$$

Thus, $\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$.

Exercise 5.

1. As the function f is 2π-periodic and odd, we conclude that

$$\int_0^{2\pi} f(y)\, dy = \int_{-\pi}^{\pi} f(y)\, dy = 0,$$

and then F is 2π-periodic. We can also show that F is odd. Indeed,

$$F(-x) = \int_0^{-x} f(y)\, dy = -\int_0^{x} f(-z)\, dz = \int_0^{x} f(z)\, dz = F(x),$$

where we used the change of variable $z = -y$, and the fact that f is odd, In addition,

$$F(x) = \int_0^x y(\pi - y)\, dy = \pi\frac{x^2}{2} - \frac{x^3}{3}, \quad x \in [0, \pi].$$

2. Using the Theorem 14.6 of the book (check also Theorem 2 in section 4.3.3 of the course notes) about integration of Fourier series.

$$
\begin{aligned}
F(x) = \int_0^x f(y)\,dy &= \int_0^x \frac{a_0^f}{2}\,dy + \sum_{n=1}^{\infty} \int_0^x \left[a_n^f \cos(ny) + b_n^f \sin(ny)\right]\,dy \\
&= \frac{a_0^f}{2} x + \sum_{n=1}^{\infty} \left[\frac{a_n^f}{n}\sin(nx) - \frac{b_n^f}{n}\cos(nx) + \frac{b_n^f}{n}\right] \\
&= \frac{8}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} - \frac{8}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}\cos\left((2n+1)x\right),
\end{aligned}
$$

since the Fourier coefficients of the function f are given by

$$
a_n^f = 0, \quad b_{2n}^f = 0, \quad \text{and} \quad b_{2n+1}^f = \frac{8}{\pi(2n+1)^3}, \quad \text{for every } n \geq 0.
$$

For finding the constant term, we proceed int he following way: because F is even

$$
\begin{aligned}
a_0^F &= \frac{2}{2\pi}\int_{-\pi}^{\pi} F(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} F(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]\mathrm{d}x \\
&= \frac{2}{\pi}\left(\frac{\pi^4}{6} - \frac{\pi^4}{12}\right) = \frac{\pi^3}{6}.
\end{aligned}
$$

Thus, the Fourier series of F is given by

$$
\frac{\pi^3}{12} - \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\cos\left((2n+1)x\right).
$$

Remarque: Using the point 4 of Execise 4, we obtain

$$
\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^3}{12},
$$

and conclude that $a_0^F = \pi^3/6$.