EPFL - Fall 2024	Dr. Pablo Antolin
Analysis III - 202 (c) GM EL MX	**Correction**
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Exercise 1.
Reminder: the *flux* of a regular orientable vector field F through the regular surface $\Sigma \subset \mathbb{R}^3$ is defined by:

$$\iint_{\Sigma} F \cdot \mathrm{d}S.$$

The sign of this expression is ambiguous if the direction of the continuous normal field (which exists since Σ is orientable) along Σ used is not specified.

Here, we have specified the choice of a unit normal field ν along Σ.

Let $\sigma : \bar{A} \mapsto \Sigma$ be a parameterization of Σ defined by $\sigma(u, v)$, **which respects the choice of orientation of** Σ, so that

$$\frac{\sigma_u \wedge \sigma_v}{\|\sigma_u \wedge \sigma_v\|} = \nu(u, v).$$

Using the definitions from the course for surface integrals of a scalar field and a vector field, we have:

$$\begin{aligned}\iint_{\Sigma} (F \cdot \nu)\, \mathrm{d}S &= \iint_{A} [F(\sigma(u,v)) \cdot \nu(u,v)] \|\sigma_u \wedge \sigma_v\| \,\mathrm{d}u\mathrm{d}v \\ &= \iint_{A} [F(\sigma(u,v)) \cdot \sigma_u \wedge \sigma_v] \,\mathrm{d}u\mathrm{d}v \\ &= \iint_{\Sigma} F \cdot \mathrm{d}S.\end{aligned}$$

It should be noted that the sign of a surface integral of a scalar field is not ambiguous (we take the norm of $\sigma_u \wedge \sigma_v$, its direction does not matter). Here, and as in the divergence theorem, it is the explicit choice of a normal unit ν that fixes the sign.

Exercise 8.

By the divergence theorem, the quantity

$$\iint_S F_{\alpha,\beta} \cdot \mathrm{d}S$$

is equal to

$$\iiint_{B_1} \operatorname{div} F_{\alpha,\beta} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

up to a sign. (where B_1 is the ball of radius 1). Moreover,

$$\operatorname{div} F_{\alpha,\beta}(x,y,z) = \frac{1}{(y^2+z^2)^\alpha} + 1 + \frac{1}{|x|^\beta}.$$

Thus, we only need to determine for which values of α and β we obtain

$$\iiint_{B_1} \frac{1}{(y^2+z^2)^\alpha} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z < +\infty \quad \text{and} \quad \iiint_{B_1} \frac{1}{|x|^\beta} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z < +\infty. \tag{1}$$

Now note that $B_1 \subset \{(x,y,z) \in \mathbb{R}^3 : y^2+z^2 < 1\}$ and thus, since the function $\frac{1}{(y^2+z^2)^\alpha}$ is always positive, we have

$$\begin{aligned}
\iiint_{B_1} \frac{1}{(y^2+z^2)^\alpha} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z &\le \iiint_{\{(x,y,z)\in\mathbb{R}^3 : y^2+z^2<1\}} \frac{1}{(y^2+z^2)^\alpha} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\le 2 \iint_{\{(y,z)\in\mathbb{R}^3 : y^2+z^2<1\}} \frac{1}{(y^2+z^2)^\alpha} \, \mathrm{d}y\mathrm{d}z \\
&= 2 \int_0^{2\pi} \int_0^1 \frac{1}{r^{2\alpha}} r \, dr d\theta \\
&= 4\pi \int_0^1 r^{1-2\alpha} \, dr.
\end{aligned}$$

This last integral is finite if and only if $\alpha < 1$. Similarly, for the second integral of (1), we have $B_1 \subset \{(x,y,z) \in \mathbb{R}^3 : |x|,|y|,|z| < 1\}$. Since the function $\frac{1}{|x|^\beta}$ is positive, we get

$$\begin{aligned}
\iiint_{B_1} \frac{1}{|x|^\beta} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z &\le \iiint_{\{(x,y,z)\in\mathbb{R}^3 : |x|,|y|,|z|<1\}} \frac{1}{|x|^\beta} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\le 4 \int_{-1}^1 \frac{1}{|x|^\beta} \, \mathrm{d}x \\
&= 8 \int_0^1 s^{-\beta} \, ds
\end{aligned}$$

Here the last integral is finite if and only if $\beta < 1$. We conclude that

$$\left| \iint_S F_{\alpha,\beta} \, \mathrm{d}S \right| < +\infty$$

if $\alpha < 1$ and $\beta < 1$.

Remark.
We have not shown that
$$\left|\iint_S F_{\alpha,\beta}\,\mathrm{d}S\right| < +\infty$$
if and only if $\alpha < 1$ and $\beta < 1$. We have only shown that if $\alpha < 1$ and $\beta < 1$, then the above integral is finite. However, it is also true that if

$$\left|\iint_S F_{\alpha,\beta}\,\mathrm{d}S\right| < +\infty,$$

then $\alpha < 1$ and $\beta < 1$. To show this, it is sufficient to prove that if $\alpha \geq 1$ or $\beta \geq 1$, then
$$\left|\iint_S F_{\alpha,\beta}\,\mathrm{d}S\right| = +\infty. \tag{2}$$
Suppose $\alpha \geq 1$ (the case $\beta \geq 1$ is similar). By taking a cylinder (with small radius and small height) aligned with the Ox axis such that it is contained within B_1, we find (2) using the same arguments as above. This is nevertheless not the goal of the exercise.