Exercise 4.

1. D_1 is the triangle with vertices $(0,0)$, $(0,1)$, and $(1,0)$. We can parameterize it as $D_1 = \left\{(x,y) \in \mathbb{R}^2 : x \in [0,1],\, y \in [0, 1-x]\right\}$.

$$\iint_{D_1} \sqrt{1-x-y}\,\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^{1-x} \sqrt{1-x-y}\,\mathrm{d}y$$
$$= \int_0^1 \left(-\frac{2}{3}(1-x-y)^{3/2}\Big|_{y=0}^{y=1-x} \right) \mathrm{d}x$$
$$= \frac{2}{3}\int_0^1 (1-x)^{3/2}\,\mathrm{d}x = \frac{2}{3} \cdot \left(-\frac{2}{5}\,(1-x)^{5/2}\Big|_{x=0}^{x=1} \right)$$
$$= \frac{2}{3} \cdot \frac{2}{5} = \frac{4}{15}.$$

2. To parameterize D_2, we use polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, with $\theta \in [0, 2\pi]$ and $r \in [0, 2(1+\cos\theta)]$. Indeed, observe that

$$0 \le x^2 + y^2 \le 2\left(x + \sqrt{x^2+y^2}\right) \Leftrightarrow 0 \le r \le 2(1+\cos\theta).$$

Recall that for polar coordinates, the Jacobian is r.

$$\iint_{D_2} \frac{\mathrm{d}x\mathrm{d}y}{(x^2+y^2)^{\frac{3}{4}}} = \int_0^{2\pi} \mathrm{d}\theta \int_0^{2(1+\cos\theta)} \frac{1}{r^{3/2}}\, r\,\mathrm{d}r = \int_0^{2\pi} 2\sqrt{2(1+\cos\theta)}\,\mathrm{d}\theta$$

using the trigonometric identity $1+\cos\theta = 2\cos^2(\theta/2)$, we find

$$= 4\int_0^{2\pi} |\cos(\theta/2)|\,\mathrm{d}\theta = 4\int_0^{\pi} \cos(\theta/2)\,\mathrm{d}\theta - 4\int_{\pi}^{2\pi} \cos(\theta/2)\,\mathrm{d}\theta$$
$$= 8\,\sin(\theta/2)\big|_{\theta=0}^{\theta=\pi} - 8\,\sin(\theta/2)\big|_{\theta=\pi}^{\theta=2\pi} = 16.$$

3. Finally, $D_3 = \left\{(x,y,z) \in \mathbb{R}^3 : x \in [0,1],\, y \in [0,1-x],\, z \in \left[0, 1-y^2\right]\right\}$.

$$\iiint_{D_3} z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{1-y^2} z\,\mathrm{d}z = \frac{1}{2}\int_0^1 \mathrm{d}x \int_0^{1-x} (1-y^2)^2\,\mathrm{d}y$$
$$= \frac{1}{2}\int_0^1 \left(y - \frac{2}{3}y^3 + \frac{1}{5}y^5 \Big|_{y=0}^{y=1-x} \right) \mathrm{d}x$$
$$= \frac{1}{2}\int_0^1 \left((1-x) - \frac{2}{3}(1-x)^3 + \frac{1}{5}(1-x)^5 \right) \mathrm{d}x$$
$$= \frac{1}{2}\left(-\frac{1}{2}(1-x)^2 + \frac{1}{6}(1-x)^4 - \frac{1}{30}(1-x)^6 \right)\Big|_{x=0}^{x=1} = \frac{11}{60}.$$

Exercise 5.
For the two domains D_1 and D_2, we use cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$.

1. The paraboloid, obtained by rotating around the Oz axis of the curve $z = r^2$, intersects the sphere, obtained by rotating around the Oz axis of the curve $r^2 + z^2 = 1$, when $z^2 + z - 1 = 0$ and $z > 0$, hence at $z^* = \frac{\sqrt{5}-1}{2}$. The domain $D_1 = P \cup B$ is therefore composed of a piece of paraboloid and a piece of sphere

$$P = \left\{(r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 : z \in [0, z^*],\, \theta \in [0, 2\pi],\, r \in \left[0, \sqrt{z}\right]\right\},$$
$$B = \left\{(r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 : z \in [z^*, 1],\, \theta \in [0, 2\pi],\, r \in \left[0, \sqrt{1-z^2}\right]\right\}.$$

Its volume is given by

$$\mathrm{Volume}(D_1) = \mathrm{Volume}(P) + \mathrm{Volume}(B) = \iiint_P \mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint_B \mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$= \int_0^{\frac{\sqrt{5}-1}{2}} \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{z}} r\,\mathrm{d}r + \int_{\frac{\sqrt{5}-1}{2}}^{1} \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{1-z^2}} r\,\mathrm{d}r$$
$$= 2\pi \int_0^{\frac{\sqrt{5}-1}{2}} \frac{z}{2}\,\mathrm{d}z + 2\pi \int_{\frac{\sqrt{5}-1}{2}}^{1} \frac{1-z^2}{2}\,\mathrm{d}z = \frac{5\pi}{12}\left(3 - \sqrt{5}\right).$$

2. For the domain D_2, the condition $x^2 + y^2 \leq 1$ translates to $0 \leq r \leq 1$. The two conditions $x \geq 0$ and $y \geq 0$ translate to $0 \leq \theta \leq \pi/2$. The last

conditions $z \geq 0$ and $x+y+z \leq \sqrt{2}$ become $0 \leq z \leq \sqrt{2}-r\cos\theta-r\sin\theta$. Note, however, that for all $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/2$, the quantity $\sqrt{2}-r\cos\theta-r\sin\theta \geq 0$, with equality if and only if $r=1$ and $\theta=\pi/4$. Thus,

$$D_2 = \left\{(r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 : r \in [0,1],\ \theta \in [0,\pi/2],\ z \in \left[0, \sqrt{2}-r\cos\theta-r\sin\theta\right]\right\}.$$

Its volume is given by

$$\begin{aligned}
\mathrm{Volume}(D_2) &= \iiint\limits_{D_2} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int\limits_0^1 r\,\mathrm{d}r \int\limits_0^{\pi/2} \mathrm{d}\theta \int\limits_0^{\sqrt{2}-r\cos\theta-r\sin\theta} \mathrm{d}z \\
&= \int\limits_0^1 r\,\mathrm{d}r \int\limits_0^{\pi/2} \left(\sqrt{2}-r\cos\theta-r\sin\theta\right)\mathrm{d}\theta \\
&= \int\limits_0^1 r\left(\frac{\sqrt{2}\pi}{2}-r-r\right)\mathrm{d}r = \frac{\sqrt{2}\pi}{4}-\frac{2}{3}.
\end{aligned}$$