EPFL - Fall 2024	Dr. Pablo Antolin
Analysis III - 202 (c) GM EL MX	**Correction**
Series 1	September 10, 2024

Exercise 1.

1. We have:
$$\begin{aligned}\frac{\partial}{\partial x_i}[\langle \boldsymbol{F}, \boldsymbol{G}\rangle] &= \frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n} F_j G_j\right] \\ &= \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left[F_j G_j\right] \\ &= \sum_{j=1}^{n} \left(\frac{\partial F_j}{\partial x_i} G_j + F_j \frac{\partial G_j}{\partial x_i}\right) \\ &= \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_i} G_j + \sum_{j=1}^{n} F_j \frac{\partial G_j}{\partial x_i},\end{aligned}$$
which leads to:
$$\frac{\partial}{\partial x_i}[\langle \boldsymbol{F}, \boldsymbol{G}\rangle] = \left\langle \frac{\partial \boldsymbol{F}}{\partial x_i}, \boldsymbol{G} \right\rangle + \left\langle \boldsymbol{F}, \frac{\partial \boldsymbol{G}}{\partial x_i} \right\rangle$$

2. Let us consider the first component. We use the subscript $(\cdot)_i$ to denote the i-th component of a given vector. We have:
$$\begin{aligned}\left(\frac{\partial}{\partial x_i}[\boldsymbol{F} \wedge \boldsymbol{G}]\right)_1 &= \frac{\partial}{\partial x_i}\left[(\boldsymbol{F} \wedge \boldsymbol{G})_1\right] \\ &= \frac{\partial}{\partial x_i}\left[F_2 G_3 - F_3 G_2\right] \\ &= \frac{\partial}{\partial x_i}\left[F_2 G_3\right] - \frac{\partial}{\partial x_i}\left[F_3 G_2\right] \\ &= \frac{\partial F_2}{\partial x_i} G_3 + F_2 \frac{\partial G_3}{\partial x_i} - \left(\frac{\partial F_3}{\partial x_i} G_2 + F_3 \frac{\partial G_2}{\partial x_i}\right) \\ &= \frac{\partial F_2}{\partial x_i} G_3 - \frac{\partial F_3}{\partial x_i} G_2 + F_2 \frac{\partial G_3}{\partial x_i} - F_3 \frac{\partial G_2}{\partial x_i} \\ &= \left(\frac{\partial \boldsymbol{F}}{\partial x_i}\right)_2 G_3 - \left(\frac{\partial \boldsymbol{F}}{\partial x_i}\right)_3 G_2 + F_2 \left(\frac{\partial \boldsymbol{G}}{\partial x_i}\right)_3 - F_3 \left(\frac{\partial \boldsymbol{G}}{\partial x_i}\right)_2 \\ &= \left(\frac{\partial \boldsymbol{F}}{\partial x_i} \wedge \boldsymbol{G}\right)_1 + \left(\boldsymbol{F} \wedge \frac{\partial \boldsymbol{G}}{\partial x_i}\right)_1 \\ &= \left(\frac{\partial \boldsymbol{F}}{\partial x_i} \wedge \boldsymbol{G} + \boldsymbol{F} \wedge \frac{\partial \boldsymbol{G}}{\partial x_i}\right)_1\end{aligned}$$
The exact same procedure holds true for the second and third components. Therefore, we show that
$$\frac{\partial}{\partial x_i}[\boldsymbol{F} \wedge \boldsymbol{G}] = \frac{\partial \boldsymbol{F}}{\partial x_i} \wedge \boldsymbol{G} + \boldsymbol{F} \wedge \frac{\partial \boldsymbol{G}}{\partial x_i}.$$

Exercise 2.

1. Polar coordinates:

$$\mathrm{Jac}(u)(r,\theta) = \det \nabla u(r,\theta) = \det \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = r.$$

2. Spherical coordinates:

$$\begin{aligned} \mathrm{Jac}(u)(r,\theta,\varphi) &= \det \nabla u(r,\theta,\varphi) \\ &= \det \begin{pmatrix} \cos(\theta)\sin(\varphi) & -r\sin(\theta)\sin(\varphi) & r\cos(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) & r\cos(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) \\ \cos(\varphi) & 0 & -r\sin(\varphi) \end{pmatrix} \\ &= -r^2\sin(\varphi). \end{aligned}$$

3. Cylindrical coordinates:

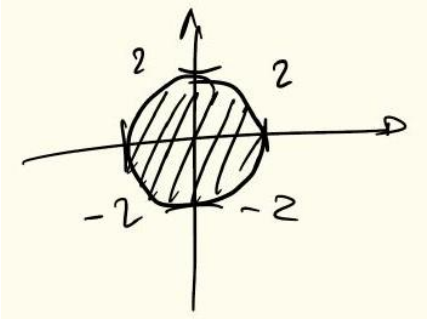
$$\mathrm{Jac}(u)(r,\theta,z) = \det \nabla u(r,\theta,z) = \det \begin{pmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = r.$$

4. Cartesian coordinates:

$$\mathrm{Jac}(u)(x,y,z) = \det \nabla u(x,y,z) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1.$$

Exercise 3.

1. The sketch:



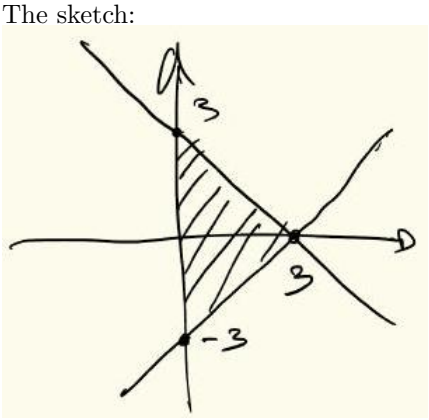
Using the polar coordinate system leads to:

$$A = \{(r\cos\theta, r\sin\theta) \mid r \in [0,2], \theta \in [0,2\pi]\},.$$

Therefore, we can evaluate the integral as follows:

$$\begin{aligned}\int_A f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \int_0^2 \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r \mathrm{d}\theta \mathrm{d}r \\ &= \int_0^2 \int_0^{2\pi} \frac{1}{r} r \mathrm{d}\theta \mathrm{d}r \\ &= 4\pi.\end{aligned}$$

2. The sketch:



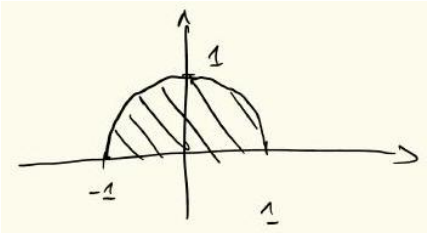
Here, we can use a Cartesian coordinate system directly:

$$\begin{aligned}\int_A f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \int_0^3 \int_{x-3}^{3-x} f(x,y)\mathrm{d}y\mathrm{d}x \\ &= \int_0^3 \int_{x-3}^{3-x} x^2 + \sin^3(y)\mathrm{d}y\mathrm{d}x \\ &= \int_0^3 (6-2x)x^2\mathrm{d}x + \int_0^3 \underbrace{\int_{x-3}^{3-x} \sin^3(y)\mathrm{d}y}_{=0}\,\mathrm{d}x.\end{aligned}$$

The second integral is zero because the integrand is a odd function and the integration domain is symmetric with respect to the origin. Thus:

$$\begin{aligned}\int_A f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \int_0^3 6x^2 - 2x^3\mathrm{d}x \\ &= \left[2x^3 - \frac{1}{2}x^4\right]_{x=0}^{x=3} \\ &= \frac{27}{2}.\end{aligned}$$

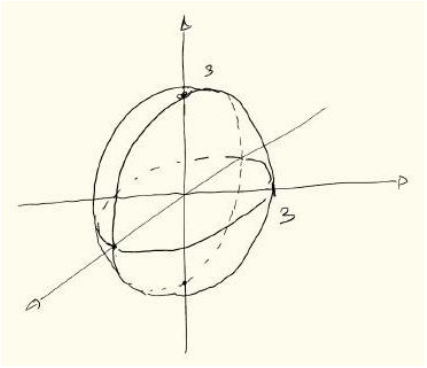
3. The sketch:



We can either use a Cartesian or a Polat coordinate system here. With Cartesian coordinates, we get:

$$\begin{aligned}\int_A f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} y\left(1+x^2\right)\mathrm{d}y\mathrm{d}x \\ &= \int_{-1}^{1}\left(1+x^2\right)\left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{1-x^2}}\mathrm{d}x \\ &= \frac{1}{2}\int_{-1}^{1}\left(1+x^2\right)\left(1-x^2\right)\mathrm{d}x \\ &= \frac{1}{2}\int_{-1}^{1} -x^4+1\mathrm{d}x \\ &= \frac{1}{2}\left[-\frac{1}{5}x^5+x\right]_{x=-1}^{x=1} = \frac{4}{5}.\end{aligned}$$

4. The sketch:



It is preferable to use the Spherical coordinate system here:

$$A = \{(r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi) \mid r\in[0,3], \theta\in[0,2\pi], \varphi\in[0,\pi]\},$$

which leads to:

$$\begin{aligned}\int_A f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \int_0^3 \int_0^{2\pi} \int_0^{\pi} r\sin\varphi r^2 \sin\varphi \mathrm{d}\varphi\mathrm{d}\theta\mathrm{d}r \\ &= 2\pi \int_0^3 r^3 dr \int_0^{\pi} \sin^2\varphi \mathrm{d}\varphi.\end{aligned}$$

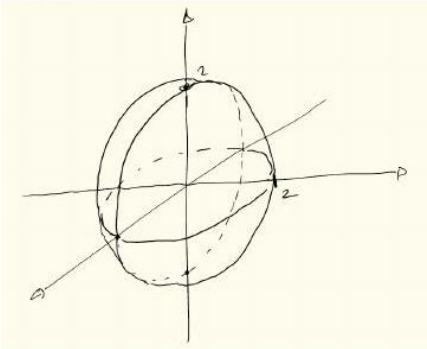
Let us remind that:

$$\frac{d}{dt}\left[\frac{1}{2}(t - \sin(t)\cos(t))\right] = \sin^2(t),$$

and therefore:

$$\begin{aligned}\int_A f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= 2\pi \left[\frac{1}{4}r^4\right]_{r=0}^{r=3} \left[\frac{1}{2}(\varphi - \sin(\varphi)\cos(\varphi))\right]_{\varphi=0}^{\varphi=\pi} \\ &= \frac{81}{4}\pi^2.\end{aligned}$$

5. The sketch:



We employ the spherical coordinate system which leads to define the domain of integration as:

$$A = \{(r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi) \mid r \in [0,2], \theta \in [0,2\pi], \varphi \in [0,\pi]\}.$$

We obtain:

$$\begin{aligned}\int_A f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \int_0^2 \int_0^{2\pi} \int_0^{\pi} \arccos(\cos\varphi) r^2 \sin\varphi \mathrm{d}\varphi\mathrm{d}\theta\mathrm{d}r \\ &= 2\pi \int_0^2 r^2 \mathrm{d}r \int_0^{\pi} \varphi\sin\varphi \mathrm{d}\varphi \\ &= 2\pi \left[\frac{1}{3}r^3\right]_{r=0}^{r=2} \left([\varphi(-\cos\varphi)]_{\varphi=0}^{\varphi=\pi} + \int_0^{\pi} \cos\varphi \mathrm{d}\varphi\right) \\ &= \frac{16}{3}\pi^2.\end{aligned}$$

Exercise 4.

1. In a first step, we can show that:

$$\nabla(|x|) = \frac{x}{|x|}, \quad \forall x \in \mathbb{R}^n \backslash \{0\}.$$

Consequently, we have:

$$\nabla\left(|x|^p\right) = p|x|^{p-1}\nabla(|x|) = p|x|^{p-2}x.$$

Remark: this function is well defined at $x = 0$ if $p \geq 2$.

2. By defining,

$$\begin{aligned} &h_p : \mathbb{R}^n \to \mathbb{R} : u \mapsto h_p(u) \\ &G : \mathbb{R}^n \to \mathbb{R}^n : v \mapsto G(v) \\ &\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^n : t \mapsto \boldsymbol{\gamma}(t) = tx \end{aligned}$$

we can rewrite the function f_p as a composition of these functions:

$$f_p = \frac{1}{p} h_p \circ \boldsymbol{G} \circ \boldsymbol{\gamma}.$$

Therefore, by applying the chain rule of differentiation, we have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f_p(t) &= \frac{1}{p} \sum_{i=1}^{n} \frac{\partial h_p}{\partial u_i}(\boldsymbol{G}(\boldsymbol{\gamma}(t))) \frac{dG_i \circ \boldsymbol{\gamma}}{dt}(t) \\ &= \frac{1}{p} \sum_{i=1}^{n} \frac{\partial h_p}{\partial u_i}(\boldsymbol{G}(\boldsymbol{\gamma}(t))) \sum_{j=1}^{n} \frac{\partial G_i}{\partial v_j}(\boldsymbol{\gamma}(t)) \frac{d\gamma^j}{dt}(t) \end{aligned}$$

Since,

$$\frac{\partial h_p}{\partial u_i}(u) = p|u|^{p-2}u_i, \quad \text{and} \quad \frac{d\gamma^j}{dt}(t) = x_j,$$

we get the final result:

$$\frac{\mathrm{d}}{\mathrm{d}t} f_p(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} |\boldsymbol{G}(tx)|^{p-2} G_i(tx) \frac{\partial G_i}{\partial v_j}(tx) x_j.$$