

Exercice 1.

1. c.f. cours

2. c.f. cours

3. En utilisant la *chain rule* et la définition de $\text{grad}(\cdot)$, on a

$$\begin{aligned}
\text{grad}(fg) &= \left(\frac{\partial(fg)}{\partial x_1}, \frac{\partial(fg)}{\partial x_2}, \dots, \frac{\partial(fg)}{\partial x_n} \right) \\
&= \left(\frac{\partial f}{\partial x_1}g + \frac{\partial g}{\partial x_1}f, \frac{\partial f}{\partial x_2}g + \frac{\partial g}{\partial x_2}f, \dots, \frac{\partial f}{\partial x_n}g + \frac{\partial g}{\partial x_n}f \right) \\
&= g \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right) + f \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial x_n} \right) \\
&= g \text{grad}(f) + f \text{grad}(g).
\end{aligned}$$

4. En utilisant la *chain rule* et la définition de $\text{grad}(\cdot)$ et $\text{div}(\cdot)$, on a

$$\begin{aligned}
\text{div}(fF) &= \sum_{i=1}^n \frac{\partial((fF)_i)}{\partial x_i} \\
&= \sum_{i=1}^n \left(f \frac{\partial F_i}{\partial x_i} + F_i \frac{\partial f}{\partial x_i} \right) \\
&= f \text{div}(F) + F \cdot \text{grad}(f).
\end{aligned}$$

5. En utilisant la définition du rotationnel, on obtient

$$\begin{aligned}
\text{rot}(\text{rot}(F)) &= \text{rot} \left(\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) \right) \\
&= \left(\frac{\partial^2 F_2}{\partial x_1 \partial x_2} - \frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_1}{\partial x_3^2} + \frac{\partial^2 F_3}{\partial x_1 \partial x_3}, \frac{\partial^2 F_3}{\partial x_2 \partial x_3} - \frac{\partial^2 F_2}{\partial x_3^2} - \frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial x_1 \partial x_2}, \right. \\
&\quad \left. \frac{\partial^2 F_1}{\partial x_3 \partial x_1} - \frac{\partial^2 F_3}{\partial x_1^2} - \frac{\partial^2 F_3}{\partial x_2^2} + \frac{\partial^2 F_2}{\partial x_2 \partial x_3} \right) \\
&= -\Delta F + \left(\frac{\partial^2 F_1}{\partial x_1^2}, \frac{\partial^2 F_2}{\partial x_2^2}, \frac{\partial^2 F_3}{\partial x_3^2} \right) + \left(\frac{\partial^2 F_2}{\partial x_1 \partial x_2} + \frac{\partial^2 F_3}{\partial x_1 \partial x_3}, \frac{\partial^2 F_3}{\partial x_2 \partial x_3} + \frac{\partial^2 F_1}{\partial x_1 \partial x_2}, \frac{\partial^2 F_1}{\partial x_3 \partial x_1} + \frac{\partial^2 F_2}{\partial x_2 \partial x_3} \right) \\
&= -\Delta F + \left(\frac{\partial}{\partial x_1} \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right), \frac{\partial}{\partial x_2} \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right), \frac{\partial}{\partial x_3} \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) \right) \\
&= -\Delta F + \text{grad} \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) \\
&= -\Delta F + \text{grad}(\text{div}(F)).
\end{aligned}$$

6. En utilisant la définition du rotationnel et du produit vectoriel ($\wedge$), on obtient

$$\begin{aligned}
 \operatorname{rot}(fF) &= \left(\frac{\partial(fF)_3}{\partial x_2} - \frac{\partial(fF)_2}{\partial x_3}, \frac{\partial(fF)_1}{\partial x_3} - \frac{\partial(fF)_3}{\partial x_1}, \frac{\partial(fF)_2}{\partial x_1} - \frac{\partial(fF)_1}{\partial x_2} \right) \\
 &= \left(f \frac{\partial F_3}{\partial x_2} + F_3 \frac{\partial f}{\partial x_2} - f \frac{\partial F_2}{\partial x_3} - F_2 \frac{\partial f}{\partial x_3}, f \frac{\partial F_1}{\partial x_3} + F_1 \frac{\partial f}{\partial x_3} - f \frac{\partial F_3}{\partial x_1} - F_3 \frac{\partial f}{\partial x_1}, \right. \\
 &\quad \left. f \frac{\partial F_2}{\partial x_1} + F_2 \frac{\partial f}{\partial x_1} - f \frac{\partial F_1}{\partial x_2} - F_1 \frac{\partial f}{\partial x_2} \right) \\
 &= f \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) + \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) \wedge (F_1, F_2, F_3) \\
 &= f \operatorname{rot}(F) + \operatorname{grad}(f) \wedge (F).
 \end{aligned}$$

Exercice 2.

Les points 1. et 2. sont vrais.

Le point 3. est faux. En effet, si $T(x_1, x_2, x_3) = e^{\frac{x_1^2}{2}} + C$ et $v = (x_1, 0, 0)$, on a

$$-\operatorname{div}(\operatorname{grad}(T)) + \operatorname{div}(Tv) = -e^{\frac{x_1^2}{2}}(1 + x_1^2) + e^{\frac{x_1^2}{2}}(1 + x_1^2) + C.$$

T ne satisfait donc l'équation de la chaleur que si $C = 0$.

Exercice 3.

1. Vrai.

2. Faux (la condition $\operatorname{div}(v) = 0$ n'est pas vérifiée); il faudrait définir $v(x_1, x_2, x_3) = (\frac{x_2^2 + x_3^2}{4}, 0, 0)$ pour que v, p satisfassent (2)-(5).

Exercice 4.

1. Vrai.

2. Faux; les équation (6)-(8) peuvent s'écrire $-\mu \operatorname{rot}(\operatorname{rot}(u)) + (\lambda + 2\mu) \operatorname{grad}(\operatorname{div}(u)) + f = 0$.

3. Vrai.

4. Vrai.

5. Vrai.