MATH-111(en) Fall 2023
Linear Algebra Annina Iseli

Revision of Topics

Section 1: Linear equations and linear maps (Weeks 1-3)

1.1 **Systems of Linear Equations** (Week 1)

1.2 **Row reduction and echelon form**
(incl. free variables and parametric form)

1.3 **Vector equations and vector arithmetic** (Week 1)

1.4 **Matrix-vector equations** (Week 2)

1.5 **The structure of the solution space**
(incl. homogenous vs. non-homogeneous equations)

1.6 **Linear combinations and linear independence** (Week 2)

1.7 **Matrices and linear maps** (Week 3)
(incl. images of unit square under linear maps;
injectivity, surjectivity, bijectivity for linear maps)

Section 2: Matrix algebra (Weeks 3-5)

2.1 **Matrix operations** (Week 3-4)
(incl. definition of matrix-matrix multiplication; matrix-matrix equations; triangular and diagonal matrices; transpose of a matrix)

2.2 **The inverse of a matrix** (Week 4-5)
(incl. expressing elementary operations as matrix multipication.)

2.3 **Characterizations of invertible matrices** (week 5)
(incl. the "endless" theorem 2.7)

Section 3: The determinant (Weeks 5-6)

3.1 **Definition of the determinant** (Week 5)

3.2 **Properties of the determinant** (Week 5-6)
(incl. determinants of elementary matrices; connection between invertibility and det)

3.3 **Computation of the determinant** (Week 5)
(incl. expansion along a column or line; and computation by row and column reduction)

3.4 **Determinants and volume** (Week 6)
(i.e. geometric interpretation of det)

Section 4: Vector spaces (Weeks 6-9)

4.1 **Definition and properties of vector spaces** (Week 6)
(incl. definition and properties of subspace of a vectorspace; definition of generating set of a space and span of vectors)

4.2 **Kernel and Range** (Week 7)
(incl. definition of the polynomial spaces $\mathbb{P}$ and $\mathbb{P}_n$)

4.3 **Linearly independent sets and bases** (Week 7-8)
(incl. how to find bases for range and kernel of a linear map)

4.4 **Coordinate systems** (Week 8)
(in particular: identifying a vector space V with $\mathbb{R}^n$ by expressing vectors $v \in V$ in terms of a basis $\mathcal{B}$ of V: $[v]_{\mathcal{B}} \in \mathbb{R}^n$.)

4.5 **The dimension of a vector space** (Week 8)

4.6 **Subspaces of finite dimensional spaces** (Weeks 8-9)
(incl. the basis expansion theorem; def. of rank; dimension formula for Ker and Ran)

4.7 **Change of basis** (Week 9)

Section 5: Eigenvalues and eigenvectors (Weeks 10-)

5.1 **Introduction to complex numbers** (Week 10)

5.2 **Eigenvalues and eigenvectors** (Week 10)
(incl. definition of similarity of matrices; characteristic polynomial)

5.3 **Diagonalization** (Week 11)

Section 6: Orthogonality and least squares (Weeks 11-13)

6.1 **The Euclidean inner product** (Week 11)
(incl. orthogonal complements)

6.2 **Orthogonal sets of vectors** (Weeks 11-12)
(incl. orthonormal sets of vectors and orthogonal matrices)

6.3 **Orthogonal projections** (Week 12)
(incl. orthogonal projections are closest-point projections)

6.4 **The Gram-Schmidt process** (Week 12)
(incl. QR-factorization)

6.5 **Least squares** (Week 13)
(incl. least square solutions, least square error)

6.6 **Application to linear models** (Week 13)
(incl. approximation of a set of data by a line with minimal least square error)

Section 7: Symmetric matrices and SVD (Weeks 13-14)

7.1 **Symmetric matrices** (Week 13)
(incl. orthogonal diagonalizability)

7.2 **Singular value decomposition (SVD)** (Week 13)

7.3 **The matrix exp(At) and systems of linear ODEs** (Week 14)

Important algorithms and computational procedures

Section 1:

- Bringing a matrix in (reduced) row echelon form (REF) or reduced row echelon form
- Finding the solution space of a linear system or a matrix-vector equation
- Writing the solution space in parametric form
- Proving linear independence of a set of vectors (by definition and/or by using theorems)
- Proving surjectivity/injectivity/bijectivity of a linear map (and in particular of a linear map that is given in matrix form. You may also use techniques from later sections.)

Section 2:

- Computing the inverses of a (square!) matrix
- Solving matrix-matrix equations
- Deciding whether a (square!) matrix is invertible
(Many tools available! E.g. the determinant or Thm.2.7 and its continuations)

Section 3:

- Computing the determinant of a (square!) matrix by various methods
(e.g. by definition, by row/column expansion (Laplace), by row/column reduction)

Section 4:

- Checking whether a given set with two operations is a vector space
- Checking whether a given subset of a vector space is a subspace
- Checking whether a given subset of a vector space is a subspace
- Finding a basis for a space/subspace
- Computing the dimension of a space/subspace (e.g. by finding a basis or by rank theorem)
- Computing a change of basis matrix $\mathcal{P}_{B\leftarrow C}$.
- Finding the kernel and range of a linear map. Finding the column space and row space of a matrix. Finding bases for each of them.
- Expressing a given linear function (resp. matrix) $f : V \to W$ in terms of a given bases $\mathcal{B}$ of V and $\mathcal{C}$ of W.

- Expressing a given linear function (resp. matrix) $f : \mathbb{R}^n \to \mathbb{R}^m$ in terms of given bases $\mathcal{B}$ of $\mathbb{R}^n$ and $\mathcal{C}$ of $\mathbb{R}^m$. (This is in some sense a special case of the previous bullet point.)

Section 5:

- Computing the characteristic polynomial, the eigenvalues (real and complex), and the eigenvectors (only those for real eigenvalues!) of a matrix
  (In other words: compute the spectrum of a matrix and compute the eigenspace of each real element of the spectrum.)
- Fiding the algebraic and the geometric multiplicty of an eigenvalue
- Deciding whether two matrices are similar.
- Deciding whether a given (square!) matrix is diagonalizable.
  ($A \in \mathbb{R}^{n\times n}$ is diagonalizable if and only if it has an eigenbasis.)
- Diagonalizing a (diagonalizable!) matrix A: Find D diagonal and P invertible so that $A = PDP^{-1}$.

Section 6:

- Finding the orthogonal complement of a subspace of a vector space ($W \subset V$ subspace: find $W^\perp$).
- Deciding whether a given set of vectors is orthogonal/orthonormal
- Deciding whether a given matrix is orthogonal (that is, by definition, if and only if its columns are orthonormal).
- Computing the orthogonal projection of a vector $v \in V$ onto a subspace $W \subset V$
- Applying Gram-Schmidt to a set of vectors (to make them orthogonal) and understanding the consequences of that.
- Computing the QR-factorization of a matrix $A \in \mathbb{R}^{m\times n}$ that has independent columns.
- Compute the least square solution(s) of a matrix vector equation and computing the least square error.
  (In particular for a matrix with $m > n$.)
- Computing the approximation of a given set of data in terms of least squares.

Section 7:

- Deciding whether a given matrix is symmetric
  (e.g. by definition $A = A^T$, or, by using orthogonal diagonalizability)
- Finding an orthogonal diagonalization of a symmetric matrix $A \in \mathbb{R}^{n\times n}$: Find D diagonal and Q orthogonal so that $A = QAQ^T$.
  (Remember: A symmetric if and only if it is orthogonally diagonalizable. This is also equivalent to A having n real eigenvalues counted with algebraic multiplicity.)
- Finding a singluar value decomposition of a matrix $A \in \mathbb{R}^{m\times n}$
- Computing $\exp(tA)$ for diagonalizable matrices
- Solving a system of linear ODEs of order 1 in cases where $\exp(tA)$ is computable (previous bullet) or is given to you.
- Solving a linear ODE of higher order by transforming it into a system of ODEs of order 1

Important examples of vector spaces

- $\mathbb{R}^n$ (and for geometric interpretations in particular $\mathbb{R}^2$ and $\mathbb{R}^3$)
- Polynomial spaces $\mathbb{P}$ and $\mathbb{P}_n$
- (Spaces of functions)
  (e.g. all functions from the real numbers to the real numbers $\{f : \mathbb{R} \to \mathbb{R}\}$)
- Spaces of matrices
  e.g. $\mathbb{R}^{m \times n}$, $\mathbb{R}^{n \times n}$, diagonal, upper (or lower) triangular, invertible, symmetric,...
- subspaces of either of the above

What to study / how to study

- Definitions and Notation
- Algorithms and standard procedures (list above)
- Examples (good heuristic: know an example and a counter-example for each definition)
- Important Theorems, Lemmas, Propositions:
  Do I understand what it is saying and how to use it?
  ("Important"= those that you often used in homework and when solving old exams; those that were not just auxiliary statements to get the next.)
- Be able to switch viewpoints (or switch notation) where applicable and helpful:
  systems of equations vs. matrices
  formulas vs. geometric interpretations
  linear maps vs. matrices
- Be fit in calculus/arithmetic e.g. with vectors, matrices, determinants, transposes, inverses,...
- For algorithms and procedures: do not only learn how to blindly execute them, but try to understand what you are doing.
- relationships between different properties of objects (e.g. invertibility, symmetry, similarity,... of matrices)
- Understand how matrices influence the properties of a set of vectors
  (e.g. If we know something about $\{v_1, v_2, v_3\}$, what do we know about $\{Av_1, Av_2, Av_3\}$? And vice-versa?)
- Ask questions! To yourself (most important!), to your peers, and to us.
- Solve problems (Homework, old exams,...)

Not part of the exam

- Gauss algorithm. (You have to be able to bring a matrix to row echelon form and/or reduced row echelon form, but you do not have to do it in the exact order prescribed by Gauss algorithm.)
- Specifics of infinite dimensional vector spaces. More precisely: you should know the definition of finite space vs. infinite space. You should know how to check the definition of vector space and subspace even in the infinite scenario (e.g. $\mathbb{P}$). But we won't ask you elaborate question on the infinite nature of infinite dimensional spaces (e.g. no questions about independence of infinite sets of vectors.).
- You will not be asked to compute the QR-decomposition of a matrix on the exam.
- Anti-symmetric matrices.
- Complex eigenvectors (understand that complex eigenvalues have complex eigenvectors. But nothing beyond that. In particular, you do not have to compute the eigenvectors for complex eigenvalues.)

Tips for proof-writing

- As a start, clearly state:
  1.) what is given (assumptions)
  2.) what we want (desired conclusion)
  Then rephrase both above bullet points into a formal statement (e.g. give the definition).
- Then try to rephrase 1.) until you end up 2.)
- If it does not work (or not at first try), then think about whether you know an equivalent way of phrasing 1.) or 2.) that might be more helpful (e.g. if you tried using $A = A^T$ for the given "A symmetric" and it did not work. Try $A = QDQ^T$ where Q orthogonal and D diagonal)
- Write things up in logical order. If necessary work on a scrap paper first, then arrange your argument in a logically consistent order, then copy to the exam sheet.
- If you rely on a specific property, mention it! (e.g. say "because A is square")
- Heuristic: If you can write up the full solution to an open problem of the exam in 2-3 lines, then likely, you should add some more reasoning. If it takes you more than a page, then likely you took an unnecessary detour (which does of course not necessarily make your solution wrong).
  Of course this is not a strict truth/rule and also depends on your hand-writing.
- You do <u>not</u> need to cite theorems by their numbering if you apply them in the exam. (Just give a brief explanation instead, e.g. "by rank theorem" or "because for square matrices injective is equivalent to surjective")
- If you get stuck or can't come up with a proof you may try to: give an example or non-example first; try to remember proofs of similar statements
- Remember: We cannot prove a statement by giving an example. But we can disprove a statement by giving an example.

Tips for the exam

- **Most importantly: be calm, focused, and confident.**
- Sleep enough the night(s) before the exam
- Sometimes taking a day of break from studying the day (or at least the evening) before the exam may be the best strategy
- **Bring a pen, an eraser (tipex), and your water bottle**
  (You must write in black, dark grey, or dark blue. If you use an additional color to highlight something in open problems, do not use red, orange or pink.)
- Plan enough time to arrive on campus.
  (e.g. take an early train so if there is a delay, it does not result in stress for you.)
- Think about how you want to handle time management!
  A good strategy might be: Start at problem one. Read through the problem. Take a minute to properly digest it. If you can answer it, then do so. If not, move on. Do this with all problems and return to the skipped ones only at the end.
- Reserve time at the end to double check your answers
  (in particular for computational problems).

This file might be updated at some point if necessary.

This version if from December 19th, 2024