Duration : 60 minutes

Linear Algebra Midterm

Fall 2024

Questions

For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 if your answer is incorrect.

For the **true/false** questions, we give

- +1 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 points if your answer is incorrect.

Notation (all standard)

– $\mathbb{R}$ denotes the set of real numbers.
– For a matrix A, $a_{ij} \in \mathbb{R}$ denotes the entry of A in row i and column j.
– For a vector $x \in \mathbb{R}^n$, x_i denotes the ith coordinate of x.
– I_m denotes the $m \times m$ identity matrix.
– $\mathbb{P}_n$ is the vector space of polynomials of degree less than or equal to n.
– $\mathbb{R}^{m\times n}$ is the vector space of $m\times n$ matrices.

First part: Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct response.

Question 1: Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \right\}$$

be two ordered bases of $\mathbb{R}^3$. Let P be the change of basis matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$ (i.e., the matrix that satisfies $[x]_{\mathcal{C}} = P[x]_{\mathcal{B}}$ for all $x \in \mathbb{R}^3$). Then, the second row of P is

☐ $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & -1 & 1 \end{pmatrix}$

☐ $\begin{pmatrix} -1 & 0 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$

Question 2: Let $\mathcal{B} = \{2-t,\ t+t^2,\ -1+t^3,\ -1-t+2t^2\}$ be an ordered basis of $\mathbb{P}_3$. The fourth coordinate of the polynomial $p(t) = t + 2t^2 + 3t^3$ with respect to the basis $\mathcal{B}$ is

☐ -7 ☐ $\frac{1}{7}$ ☐ $-\frac{1}{7}$ ☐ 3

Question 3: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x-y \\ x-y \\ -5x+6y \\ x+y \end{pmatrix}.$$

Then

☐ T is injective but not surjective ☐ T is surjective but not injective

☐ T is injective and surjective ☐ T is neither injective nor surjective

Question 4: Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 3 & 0 \\ 2 & \sqrt{3} & \pi & 3 & \sqrt{2} \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & \pi & 3 & \sqrt{2} \\ \sqrt{3} & 1 & \pi & 3 & \sqrt{2} \end{pmatrix}.$$

Then

☐ $\det(A) = 0$ ☐ $\det(A) = 12\pi$ ☐ $\det(A) = -6\pi$ ☐ $\det(A) = \sqrt{6}\pi$

Question 5: Let

$$A = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 5 & -1 \\ 1 & -1 & 2 & 2 \\ 3 & 1 & 0 & 1 \end{pmatrix}.$$

If $B = A^{-1}$ is the inverse of the matrix A, then

☐ $b_{33} = \frac{4}{39}$ ☐ $b_{41} = \frac{1}{3}$ ☐ $b_{33} = -\frac{1}{13}$ ☐ $b_{43} = \frac{2}{3}$

Question 6: Let W be the vector space of $2{\times}2$ symmetric matrices and let $T\colon \mathbb{P}_2 \to W$ be the linear transformation defined by

$$T(a + bt + ct^2) = \begin{pmatrix} a & b-c \\ b-c & a+b+c \end{pmatrix} \qquad \text{for all } a, b, c \in \mathbb{R}.$$

Let

$$\mathcal{B} = \{1,\, 1-t,\, t+t^2\} \qquad \text{and} \qquad \mathcal{C} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be ordered bases of $\mathbb{P}_2$ and W, respectively. The matrix A associated to T relative to the bases $\mathcal{B}$ of $\mathbb{P}_2$ and $\mathcal{C}$ of W (i.e., the matrix satisfying $[T(p)]_{\mathcal{C}} = A[p]_{\mathcal{B}}$ for all $p \in \mathbb{P}_2$) is

☐ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Question 7: The system of linear equations

$$\begin{cases} x_1 + 2x_2 + 5x_3 - 4x_4 = 0 \\ \phantom{x_1 +{}} x_2 + 2x_3 + x_4 = 7 \\ \phantom{x_1 +{}} x_2 + 3x_3 - 5x_4 = -4 \\ 2x_1 + 3x_2 + 4x_3 - 3x_4 = 1 \end{cases}$$

has a unique solution which satisfies

☐ $x_1 = 2$ ☐ $x_1 = -3$ ☐ $x_1 = -2$ ☐ $x_1 = 3$

Question 8: Let $t \in \mathbb{R}$ be a parameter. The vectors

$$v_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -3 \\ 5 \\ -2 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} t \\ -9 \\ 8 \end{pmatrix}$$

are linearly dependent if and only if

☐ $t = 5$ ☐ $t = -5$ ☐ $t \neq 5$ ☐ $t \neq -5$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 9 : If $A, B \in \mathbb{R}^{n\times n}$ are two invertible matrices such that $A+B$ is not the zero matrix, then $A+B$ is also invertible.

☐ TRUE ☐ FALSE

Question 10 : Let $A \in \mathbb{R}^{m\times n}$ where $m < n$. If the reduced echelon form of the matrix A has exactly k zero rows, then the set of solutions of the homogeneous system $Ax = 0$ is a subspace of $\mathbb{R}^n$ of dimension $n - k$.

☐ TRUE ☐ FALSE

Question 11 : Let $A \in \mathbb{R}^{n\times n}$ and let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation defined by $T(x) = Ax$, for all $x \in \mathbb{R}^n$. If A is such that $A^5 = 0$, then T is surjective.

☐ TRUE ☐ FALSE

Question 12 : Let V and W be two vector spaces and $T\colon V \to W$ a linear transformation.
If $\dim(\operatorname{Ker} T) = \dim V$, then $\operatorname{Ran} T = \{0_W\}$.

☐ TRUE ☐ FALSE

Question 13 : Let q be an arbitrary polynomial of degree 3. Then the set

$$\{p \in \mathbb{P}_3 : q(0) - p(0) = 0\}$$

is a subspace of $\mathbb{P}_3$.

☐ TRUE ☐ FALSE

Question 14 : Let $A \in \mathbb{R}^{4\times 4}$ be a rank 3 matrix. If u, v, w are linearly independent vectors in $\mathbb{R}^4$, then Au, Av, Aw are linearly independent vectors in $\mathbb{R}^4$.

☐ TRUE ☐ FALSE

Question 15 : If $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix, then

$$\det(A - A^T) = \det(A) - \det(A^T).$$

☐ TRUE ☐ FALSE

Question 16 : Let W be the subspace of $\mathbb{P}_5$ spanned by $p_1, p_2, p_3, p_4 \in \mathbb{P}_5$. If $\dim(W) = 4$, then there exist two polynomials $p_5, p_6 \in \mathbb{P}_5$ such that the set $\mathcal{B} = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ is a basis of $\mathbb{P}_5$.

☐ TRUE ☐ FALSE