MATH-111(en) FALL 2024
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Homework 7

Ex 7.1 (Non-subspaces of the plane)

Show that none of the following sets is a subspace of $\mathbb{R}^2$:

a) $S_1 = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0\}$;

b) $S_2 = \{(x,y) \in \mathbb{R}^2 : x \cdot y \geq 0\}$;

c) $S_3 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

(By the way, can you tell what each of these sets looks like? Try to draw them!)

Ex 7.2 (Is it a vector space?)

For each of the following sets (equipped with the obvious addition and scalar multiplication), decide whether it is a vector space and prove your result.

$$A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = 0 \right\}, \quad B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y = 1 \right\}, \quad C = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : z = y \right\}$$

$$D = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \{0, -1, 1\} \right\}, \quad E = \left\{ f : \mathbb{R}^3 \to \mathbb{R}^3 \text{ linear} : f(e_1) = 0 \right\}$$

Ex 7.3 (Spaces of polynomials)

Let P_n be the vector space of polynomials of degree less than or equal to n. Determine if each of the following sets is a subspace of P_n for a given n. (You may take for granted that P_n is a vector space.)

a) The set of polynomials of the form $p(t) = at^2$ where a is an arbitrary real number.

b) The set of polynomials of the form $p(t) = a + t^2$ where a is an arbitrary real number.

c) The set of polynomials of the form $p(t) = c_1 t^3 + c_2 t^2 + c_3 t + c_4$, where c_1, c_2, c_3 and c_4 are non-negative integers.

d) The set of polynomials in P_n that satisfy $p(0) = 0$.

Ex 7.4 (Sum of subspaces)

Let V be a vector space and let S and T be subspaces of V. Prove that:

(a) $S \cap T := \{v \in V : v \in S \text{ and } v \in T\}$ is a subspace of V

(b) $S + T := \{s + t : s \in S,\ t \in T\}$ is a subspace of V

(c) $S \cup T := \{v \in V : v \in S \text{ or } v \in T\}$ is not a subspace of V.

Ex 7.5 (A subspace of polynomials)
Let $\mathbb{P}_3$ be the vector space of polynomials $p(t)$ of degree at most 3.

(a) Let S be the subspace spanned by

$$p_1(t) = 1 + t^2, \quad p_2(t) = 3t + 4t^3, \quad p_3(t) = 1 + t + 5t^2 + 4t^3.$$

Is $1 + 2t + 3t^2 + 4t^3$ an element of S?

(b) Define $\tilde{\mathbb{P}_3}$ to be the set of polynomials of degree exactly 3. Is $\tilde{\mathbb{P}_3}$ a vector space?

Ex 7.6 (The only finite subspaces is $\{0_v\}$.)
Let V be a vector space and 0_V its zero element. Prove that $\{0_V\}$ is the only subspace of V that consists of only finitely many elements.

(While this is a fun and worthwhile problem, don't spend too much time on it if there are other problems on the homework, that you have not yet solved.)

Ex 7.7 (Column space and kernel)

(a) Does $v = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ lie in the column space of $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{pmatrix}$? Does it lie in its kernel?

(b) Let $B = \begin{pmatrix} 1 & 2 & -3 \\ 4 & -1 & 0 \\ 0 & -3 & 4 \end{pmatrix}$. Find a nonzero vector $u \in \mathrm{Col}(B)$ and a nonzero vector $v \in \mathrm{Ker}(B)$. Is there a nonzero vector that lies in both $\mathrm{Col}(B)$ and $\mathrm{Ker}(B)$?

(c) Express the kernel of the following matrix in parametric vector form:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

Ex 7.8 (Column space and kernel)
(a) Consider

$$w = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 3 & -5/2 \\ -3 & -2 & 4 \\ 2 & 4 & -4 \end{pmatrix}.$$

Find out if w is in $\mathrm{Col}\, A$, in $\mathrm{Ker}\, A$, or both.

(b) Find bases for the kernel, the column space, and the row space of $A = \begin{pmatrix} 1 & 1 & 5 & 1 \\ 2 & 4 & 14 & 4 \\ 2 & 3 & 12 & 3 \end{pmatrix}$

Ex 7.9 ((When) do linear maps preserve linear (in)dependence?)

Let V and W be two vector spaces, $T : V \to W$ a linear transformation and $\{v_1, \dots, v_p\}$ a subset of V.

1. Show that if the set $\{v_1, \dots, v_p\}$ is linearly dependent, then the set $\{T(v_1), \dots, T(v_p)\}$ is linearly dependent too.
2. Assume T is an injective transformation : $T(u) = T(v) \Rightarrow u = v$. Show that if the set $\{T(v_1), \dots, T(v_p)\}$ is linearly dependent, then the set $\{v_1, \dots, v_p\}$ is linearly dependent too.

Ex 7.10 (A subspace and a possible basis)

Let $S \subset \mathbb{R}^4$ be the subset of vectors $(x_1, x_2, x_3, x_4)^T$ satisfying the equations

$$x_1 - 2x_3 + x_4 = 0, \quad x_2 + 3x_3 = 0, \quad \text{and} \quad x_1 - x_4 = 0.$$

Show that S is a subspace of $\mathbb{R}^4$. Find a basis for S.

Ex 7.11 (The range of linear maps)

Let V, W be vector spaces and $T : V \to W$ be linear. Show that $\mathrm{Ran}(T)$ is a subspace of W.

Ex 7.12 (True/False questions)

Decide whether the following statements are always true or if they can be false.

(i) Let V be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$. Then the set of functions such that $f(3) = 0$ is a subspace.

(ii) Let V be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$. Then the set of functions such that $f(3) \cdot f(6) = 0$ is a subspace.

(iii) Let $M_{2\times 2}$ be the vector space of all 2×2 matrices, and let S be the subset of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a, b \in \mathbb{R}$. Then S is a subspace.

(iv) Let $A \in \mathbb{R}^{n\times n}$. If $\mathrm{Ker}(A) = \{0\}$ then $\mathrm{Ker}(A^2) = \{0\}$.

(v) Let $A \in \mathbb{R}^{n\times n}$. Then $\mathrm{Ker}(A) = \{0\}$ if and only if $\mathrm{Ran}(A) = \mathbb{R}^n$.

(vi) Let $A \in \mathbb{R}^{n\times n}$. Then $\mathrm{Ker}(A) = \mathbb{R}^n$ if and only if $\mathrm{Ran}(A) = \{0\}$.