MATH-111(en) FALL 2024

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Homework 6

Ex 6.1 (The determinant of triangular matrices)
Let $A \in \mathbb{R}^{n\times n}$ be a lower (resp. upper) triangular matrix. Show that the determinant of A equals the product of its diagonal elements, i.e., $\det(A) = a_{11} \cdot \ldots \cdot a_{nn} = \prod_{i=1}^{n} a_{ii}$.
Hint: Lower triangular A: use the definition of the determinant and induction on n. Upper triangular A: use that $\det(A^T) = \det(A)$.

Ex 6.2 (Row reduction and the determinant)
Let $A \in \mathbb{R}^{n\times n}$ and assume that A_1 is obtained from A by applying an elementary operation on the rows of A. Show that

$$\det(A) = \begin{cases} -\det(A_1) & \text{if two rows or columns have been swapped,} \\ \lambda^{-1}\det(A_1) & \text{if a row or column has been multiplied by } \lambda \neq 0, \\ \det(A_1) & \text{if a multiple of a row (resp. column) has been added to another row (resp. colu} \end{cases}$$

Hint: We know that $A_1 = EA$ for some elementary matrix E.

Ex 6.3 (Different methods for computing determinants)
Compute the determinant of each of the following matrices in three ways: Once using cofactor expansion across a row, once using cofactor across a column, and once with row reduction.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 0 & 8 & 0 \end{pmatrix}$$

Ex 6.4 (Determinants based on another determinant)
Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and assume that $\det(A) = 7$. Compute the determinants of the following matrices

$$B = \begin{pmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{pmatrix}, \quad C = \begin{pmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{pmatrix}.$$

Ex 6.5 (More determinants)
Compute the determinants of the following matrices (You may use your preferred method or try to practice different methods.)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 10 & 5 & 10 & 5 \\ 6 & 9 & 0 & -3 \\ 3 & 0 & 0 & 3 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Ex 6.6 (Determinant of an antidiagonal matrix)
Find the determinant of the following *antidiagonal* matrix:

$$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}$$

What is the determinant of an $n \times n$ antidiagonal matrix with all its antidiagonal entries equal to 2?

Ex 6.7 (Determinants and volume)

(a) Calculate the volume of the parallelepiped with the following vertices:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

(b) Calculate the area of the triangle whose vertices are the points $(1,2), (2,4), (3,3) \in \mathbb{R}^2$.

Ex 6.8 (Abstract determinant calculations)
Show that

(a) if A is an invertible matrix, then $\det(A^{-1}) = 1/\det A$,

(b) if A and P are square matrices, with P invertible, then $\det(PAP^{-1}) = \det A$,

(c) if U is a square matrix such that $U^T U = I$, then $\det U = \pm 1$,

(d) if A is a square matrix such that $\det(A^4) = 0$, then A cannot be invertible.

Ex 6.9 (Multiple choice and True/False questions)

a) Let A be an $n \times n$ matrix with nonzero determinant. Then $\det(A + A) =$
(A) 0 (B) $\det(A)$ (C) $2\det(A)$ (D) $2^n \det(A)$.

b) In the following, we assume that all the matrices involved are square matrices. Decide whether the following statements are always true or if they can be false.

(i) The following matrix is invertible.

$$\begin{pmatrix} 0 & 0 & 0 & 4 \\ 2 & 0 & 4 & 3 \\ 1 & 0 & 2 & 5 \\ 3 & 6 & 1 & 8 \end{pmatrix}$$

(ii) The (i,j)-cofactor of a square matrix A is the matrix A_{ij} obtained by deleting form A its i-th row and j-th column.

(iii) If A and B are both $n \times n$, then $\det(A + B) = \det(A) + \det(B)$.

(iv) If A and B are row equivalent, i.e., they can be obtained from each other by finitely many elementary operations, then they have the same determinant.

(v) The linear transformation associated to A is injective if and only if $\det(A) \neq 0$.

(vi) The determinant of A is the product of the pivots in any echelon form U of A, multiplied by $(-1)^r$, where r is the number of row interchanges made during row reduction from A to U.

(vii) The determinant of A is the product of the diagonal entries in A.

(viii) If $\det A$ is zero, then two rows or two columns are the same, or a row a column is zero.