Discussion of mock

Question [SCQ-02-matrice-injectivite-surjectivite-2023]**:**
Let $T\colon \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x-y \\ x-y \\ -5x+6y \\ x+y \end{pmatrix}.$$

Then

- ☐ T is injective and surjective
- ■ T is injective but not surjective
- ☐ T is neither injective nor surjective
- ☐ T is surjective but not injective

- $m = 4 > 2 = n \Rightarrow$ not surjective

- Injective?

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x - y = 0 \text{ and } x + y = 0$$
$$\Rightarrow x = -y \text{ and } x = y$$
$$\Rightarrow x = 0 = y$$

this is shorter than computing Ker T completely.

$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only element in $\mathrm{Ker}(T)$

$\Rightarrow$ T injective.

- Alternative solution:

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -5 \\ -1 \end{pmatrix}, \quad T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \\ 2 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 1 \\ -5 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 11 \\ 2 \end{pmatrix}$ indep. $\Rightarrow \dim \mathrm{Im}(A) \geq 2$

Rank-Thm $\Rightarrow \quad n = 2 = \dim \mathrm{Ker}\,T + \underbrace{\dim \mathrm{Im}\,T}_{\geq 2}$

$\Rightarrow \dim \mathrm{Ker}\,T = 0$

$\Rightarrow \mathrm{Ker}\,T = \{0\} \Rightarrow T$ injective

Question [`SCQ-06-matrice-changement-base-2023`] **:**

Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \right\}$$

be two ordered bases of $\mathbb{R}^3$. Let P be the change of basis matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$ (i.e., the matrix that satisfies $[x]_{\mathcal{C}} = P[x]_{\mathcal{B}}$ for all $x \in \mathbb{R}^3$). Then, the second row of P is

- ☐ $\begin{pmatrix} -1 & 0 & 0 \end{pmatrix}$
- ■ $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$
- ☐ $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$
- ☐ $\begin{pmatrix} 0 & -1 & 1 \end{pmatrix}$

$$P = P_C^{-1} P_B$$

$$P_B = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad P_C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix} \Rightarrow P_C^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -2 & 3 \\ -1 & 1 & -2 \end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -2 & 3 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix} \leftarrow \text{second row}$$

Question [`SCQ-07-matrice-relat-base-2023`] **:**

Let W be the vector space of 2×2 symmetric matrices and let $T\colon \mathbb{P}_2 \to W$ be the linear transformation defined by

$$T(a+bt+ct^2) = \begin{pmatrix} a & b-c \\ b-c & a+b+c \end{pmatrix} \qquad \text{for all } a,b,c \in \mathbb{R}.$$

Let

$$\mathcal{B} = \{1,\ 1-t,\ t+t^2\} \qquad \text{and} \qquad \mathcal{C} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be ordered bases of $\mathbb{P}_2$ and W, respectively. The matrix A associated to T relative to the bases $\mathcal{B}$ of $\mathbb{P}_2$ and $\mathcal{C}$ of W (i.e., the matrix satisfying $[T(p)]_{\mathcal{C}} = A[p]_{\mathcal{B}}$ for all $p \in \mathbb{P}_2$) is

☐ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix}$ ■ $\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ ☐ $\cancel{\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$

Formula : $A = \left([T(b_1)]_C \, , [T(b_2)]_C \, , [T(b_3)]_C \right)$

$T(b_1) = T(1 + 0 \cdot t + 0 \cdot t^2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \cdot c_1 + 1 \cdot c_3$

$\Rightarrow [T(b_1)]_C = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$T(b_2) = T(1 - t + 0 \cdot t^2) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} = 1 \cdot c_1 - c_2$

$\Rightarrow [T(b_2)]_C = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Question [SCQ-06-lin-indep-2022]**:**

Let $t \in \mathbb{R}$ be a parameter. The vectors

$$v_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -3 \\ 5 \\ -2 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} t \\ -9 \\ 8 \end{pmatrix}$$

are linearly dependent if and only if

☐ $t = -5$ ■ $t = 5$ ☐ $t \neq 5$ ☐ $t \neq -5$

v_1, v_2, v_3 indep

$$\Leftrightarrow \underbrace{\lambda_1 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix} + \lambda_3 \begin{pmatrix} t \\ -9 \\ 8 \end{pmatrix} = 0}_{\text{Can be written as: } \underbrace{\begin{pmatrix} 1 & -3 & t \\ -2 & 5 & -9 \\ 3 & 2 & 8 \end{pmatrix}}_{=:A} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \quad \underline{\text{only}} \text{ has the solution } \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Leftrightarrow$ $Ax = 0$ has a unique solution

<u>Solution 1</u> (somewhat lengthy)

Compute RREF of A and check for which t the solution space is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

You will find: that is the case only when $t = 5$.

<u>Solution 2</u> (faster)

By alphabet thm: $Ax = 0$ has a unique solution $\Leftrightarrow$ A invertible

$\Leftrightarrow \det A \neq 0$

Compute $\det(A) = 55 - 11 \cdot t$

Observe: If $t = 5 \Leftrightarrow \det(A) = 0$

$\Rightarrow v_1, v_2, v_3$ dependent if and only if $t = 5$.

Question [TF-03-independance-lineaire-2023] **:**

Let $A \in \mathbb{R}^{4\times 4}$ be a rank 3 matrix. If u, v, w are linearly independent vectors in $\mathbb{R}^4$, then Au, Av, Aw are linearly independent vectors in $\mathbb{R}^4$.

☐ TRUE ■ FALSE

$f(x) = Ax \qquad f: \mathbb{R}^4 \to \mathbb{R}^4$

$\operatorname{rank} A = 3 \Rightarrow \dim \operatorname{Ran}(f) = 3 \Rightarrow \exists$ 3 indep. vectors in $\operatorname{Ran}(A)$

$\Rightarrow \exists$ 3 independent vectors of the type Av, Aw, Au.

But there might also be some u, w, v such that

Av, Aw, Au are not independent! Are there?

Let's try to make an example:

What is your favorite matrix $A \in \mathbb{R}^{4\times 4}$ of rank $= 3$?

Choose $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Then $Ae_1 = e_1$, $Ae_2 = e_2$, $Ae_4 = 0$

So e_1, e_2, e_4 indep. but Ae_1, Ae_2, Ae_4 dependent.

Question [TF-04-sous-espace-polynome-2023] :

Let q be an arbitrary polynomial of degree 3. Then the set

$$\{p \in \mathbb{P}_3 : q(0) - p(0) = 0\} = S$$

is a subspace of $\mathbb{P}_3$.

☐ TRUE ■ FALSE

Solution 1

Let $p, r \in S$. Is $p + r \in S$?

$p, r \in S \Rightarrow \begin{cases} q(0) - p(0) = 0 \\ q(0) - r(0) = 0 \end{cases} \Rightarrow \underbrace{2q(0) - (p(0) + r(0)) = 0}_{\text{have}}$

But we want $\underbrace{q(0) - (p(0) + r(0)) = 0}_{\text{want}}$

$\leadsto$ The claim might be false!

Counter example? Choose q so that $q(0) \neq 0$

(otherwise have and want are the same)

e.g. choose $q(x) = x^3 + 1$ (q must be deg $= 3$!)

$\Rightarrow q(0) \neq 0$ so have $\neq$ want

Solution 2 (shorter)

Is $0 \in S$?

Let $p = 0$ the zero polynomial : $p \in S \iff \underbrace{p(0)}_{=0} - q(0) = 0 \iff q(0) = 0$

So in case $q(0) \neq 0$, the claim is false.

this possible: $q(x) = x^3 + 1$

Question [TF-08-matrice-injectivite-surjectivite-2023]**:**

Let $A \in \mathbb{R}^{n\times n}$ and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation defined by $T(x) = Ax$, for all $x \in \mathbb{R}^n$. If A is such that $A^5 = 0$, then T is surjective.

☐ TRUE ■ FALSE

$T : \mathbb{R}^n \to \mathbb{R}^n$, $T(x) = Ax$

T surjective if and only if T injective (since $m = n$)

Is T injective ? Assume it is injective.

$\Rightarrow Ax \neq 0 \ \forall x \in \mathbb{R}^n$

$= A^5 x \neq 0 \ \forall x \in \mathbb{R}^n$ ↯ $A^5 = 0$

$\Rightarrow$ T not injective

$\Rightarrow$ T not surjective.