Série 14

Keywords: Singular value decomposition.

Reminder: Singular values.
The singular values of $A \in M_{m\times n}$ are defined by $\sigma_i = \sqrt{\lambda_i}$ where λ_i are the eigenvalues (always positive) of A^TA. We always have

$$\sigma_i = ||A\vec{v}_i||, \quad \text{where } \vec{v}_i \text{ is a unit eigenvector of } A^TA.$$

Reminder: Singular value decomposition.
A matrix $A \in M_{m\times n}$ of rank k can always be written as $A = U\Sigma V^T$ with

- $\Sigma \in M_{m\times n}$ is the matrix of the singular values of A:

  $$\Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \in M_{m\times n}, \quad \text{with} \quad D = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{pmatrix}, \quad \sigma_1 \geq \cdots \geq \sigma_k > 0,$$

  where σ_i are the **non-zero** singular values of A.

- $V = \begin{pmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{pmatrix} \in M_{n\times n}$ is the **orthogonal** matrix of the eigenvectors of A^TA, ordered according to **the decreasing order of its eigenvalues**.

- $U = \begin{pmatrix} A\vec{v}_1 & \dots & A\vec{v}_k & \vec{u}_{k+1} & \dots & \vec{u}_m \end{pmatrix} \in M_{m\times m}$ is the **orthogonal** matrix of the image of the eigenvectors of A^TA, completed, if necessary (when $k < m$), to a basis of $\mathbb{R}^m$ with unit vectors.

Question 1 Let A be a matrix of size $m \times n$.

a) Show that $\mathrm{Ker}A = \mathrm{Ker}(A^TA)$.

b) Show that A^TA is invertible if and only if the columns of A are linearly independent.

Question 2 Find the singular value decomposition of the matrices

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Question 3 Let A be a matrix, and let $\vec{w}_1, \vec{w}_2$ be two eigenvectors of the matrix $A^T A$, such that

$$\vec{w}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \ \vec{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ A\vec{w}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \ A\vec{w}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Use this information to find matrices U, Σ, and V such that A has a singular value decomposition of the form

$$A = U\Sigma V^T.$$

Suggested approach:

- First deduce the sizes of the matrices A, U, Σ, and V;
- Normalize the vectors $\vec{w}_1$ and $\vec{w}_2$ to obtain $\vec{v}_1$ and $\vec{v}_2$;
- Compute $A\vec{v}_1$ and $A\vec{v}_2$;
- Compute the singular values and define Σ;
- Complete $\vec{v}_1$ and $\vec{v}_2$ to form a basis of $\mathbb{R}^4$ and ensure it is an orthonormal basis using the orthogonal complement method;
- Define V using $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$;
- Normalize $A\vec{v}_1$ and $A\vec{v}_2$ and use them to define U.