Séries 2

Keywords: Linear systems, augmented matrices, Gauss-Jordan algorithm, vectors, linear combinaisons.

Question 1

(1) Write the augmented matrices of the following linear systems.

(2) Solve these linear systems with elementary operations over the rows of the matrices.

a) $\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$

b) $\begin{cases} 3x_1 + 2x_2 - x_3 = 12 \\ x_3 + 2x_1 - 4x_2 = -1 \\ x_2 + 2x_3 - 4x_1 = -8 \end{cases}$

c) $\begin{cases} 6x_1 - 3x_2 + 2x_3 = 11 \\ -3x_1 + 2x_2 - x_3 = -4 \\ 5x_1 - 3x_2 + 2x_3 = 9 \end{cases}$

d) $\begin{cases} x_1 - 3x_2 = 5 \\ 5x_3 - x_1 + x_2 = 2 \\ x_2 + x_3 = 0 \end{cases}$

Solution:

a) Augmented matrix: $\left(\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right)$,

Reduced row echelon form: $\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right)$, Solution: $x_1 = 3$, $x_2 = 2$.

b) Augmented matrix: $\left(\begin{array}{ccc|c} 3 & 2 & -1 & 12 \\ 2 & -4 & 1 & -1 \\ -4 & 1 & 2 & -8 \end{array}\right)$,

Reduced row echelon form: $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$, Solution: $x_1 = 3$, $x_2 = 2$, $x_3 = 1$.

c) Augmented matrix: $\left(\begin{array}{ccc|c} 6 & -3 & 2 & 11 \\ -3 & 2 & -1 & -4 \\ 5 & -3 & 2 & 9 \end{array}\right)$,

Reduced row echelon form: $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right)$, Solution: $x_1 = 2$, $x_2 = 3$, $x_3 = 4$.

d) Augmented matrix: $\left(\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right)$,

Reduced row echelon form: $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$, Solution: $x_1 = 2$, $x_2 = -1$, $x_3 = 1$.

Question 2

(1) Put the following matrices into row echelon form, then into reduced row echelon form.

(2) Suppose these matrices are augmented matrices of linear systems. Determine in each case whether the linear system has exactly one solution, infinitely many solutions, or no solution.

$$A = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{array}\right) \qquad B = \left(\begin{array}{c|c} 1 & 3 \\ -4 & 2 \\ -3 & -2 \end{array}\right) \qquad C = \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{array}\right)$$

Solution: A: $\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$, infinitely many solutions (x_3 is a free variable),

B: $\left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}\right)$, no solution,

C: $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$, no solution.

Question 3

(1) Verify if the following matrices are in row echelon form (REF) or in reduced row echelon form (RREF).

(2) Identify the pivot (or leading) variables and the free variables.

(3) Determine if the corresponding linear systems have exactly one solution, infinitely many solutions, or no solution.

$$A = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \qquad B = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad C = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$D = \left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right) \qquad E = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

Solution:

A) REF, RREF. Pivot variables : x_1, x_2, x_3. No free variables. Unique solution.

B) REF, RREF. Pivot variables : x_1, x_2. Free variable : x_3. Infinitely many solutions

C) REF, RREF. Pivot variables : x_1, x_2. Free variable: x_3. No solution.

D) REF. Variables principales: x_2, x_3. Free variables: x_1. Infinitely many solutions

E) Not REF. Infinitely many solutions

Question 4 Determine if the following homogeneous linear systems have a non-trivial solution.

a) $$\begin{cases} 2x_1 - 5x_2 + 8x_3 = 0 \\ -2x_1 - 7x_2 + x_3 = 0 \\ 4x_1 + 2x_2 + 7x_3 = 0 \end{cases}$$

c) $$\begin{cases} -7x_1 + 37x_2 + 119x_3 = 0 \\ 5x_1 + 19x_2 + 57x_3 = 0 \end{cases}$$

b) $$\begin{cases} x_1 - 3x_2 + 7x_3 = 0 \\ -2x_1 + x_2 - 4x_3 = 0 \\ x_1 + 2x_2 + 9x_3 = 0 \end{cases}$$

Solution:

a) The coefficient matrix is square (as many equations as unknowns) and we can observe that $L_1 = L_2 + L_3$. This is a linear dependency among the rows. We will obtain a row of zeros in the matrix, and there will not be three pivots. Thus, there is at least one free variable, which means there are infinitely many (non-trivial) solutions.

We can also solve the system. The reduced row echelon form of the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{17}{8} & 0 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The solution is

$$\begin{cases} x_1 &= -\frac{17}{8}x_3 \\ x_2 &= \frac{3}{4}x_3 \end{cases}.$$

There are infinitely many non-trivial solutions (prendre $x_3 \neq 0$).

b) RREF of the augmented matrix :

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Solution triviale ($x_1 = x_2 = x_3 = 0$).

c) The system has fewer equations (two) than unknowns (three), so there cannot be three pivots. The system is consistent (we verify this by putting it into echelon form), so there are infinitely many solutions.

Question 5

For each of the following linear systems

$$\begin{cases} x + 2y &= k \\ 4x + hy &= 5 \end{cases} \qquad \begin{cases} -3x + hy &= 1 \\ 6x + ky &= -3 \end{cases}$$

find every values of h and k such that the system

(1) has no solution,

(2) has a unique solution

(3) has infinitely many solutions

Solution: For the first system, the augmented matrix is

$$\begin{array}{cc|c} 1 & 2 & k \\ 4 & h & 5 \end{array} \sim_{L_2 - 4L_1} \begin{array}{cc|c} 1 & 2 & k \\ 0 & h-8 & 5-4k \end{array}$$

(1) if $h = 8$ and $k \neq \frac{5}{4}$, there is no solution

(2) if $h \neq 8$, the solution is unique,

(3) if $h = 8$ et $k = \frac{5}{4}$, there is infinitely many solutions.

For the second system,

$$\begin{array}{cc|c} -3 & h & -1 \\ 6 & k & -3 \end{array} \sim_{2L_2+L_1} \begin{array}{cc|c} -3 & h & 1 \\ 0 & k+2h & -1 \end{array}$$

Thus

(1) if $k = -2h$, there is no solution,

(2) if $k \neq -2h$, there is a unique solution,

(3) there do not exist values of h and k such that there are infinitely many solutions.

Question 6

(1) If the coefficient matrix of a system with four equations and four unknowns has a pivot in each column, then the system is consistent.

- [ ] False
- [x] True

(2) If the coefficient matrix of a system with four equations and four unknowns has a pivot in each row, then the system is consistent.

- [ ] False
- [x] True

(3) If the **augmented** matrix of a system with four equations and four unknowns has a pivot in each row, then the system is consistent.

- [x] False
- [ ] True

(4) If the augmented matrix of a system with four equations and four unknowns has a pivot in each column, then the system is consistent.

- [x] False
- [ ] True

Solution:

(1) This is true. A pivot in each of the four columns implies the existence of a pivot in each row. This can be concluded from a result discussed in the course.

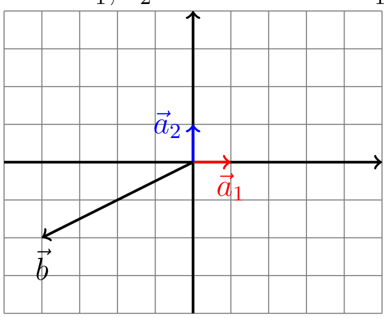
(2) This is true and is stated as such in the course.

(3) This is false. It is enough for the last row to be of the form (0 0 0 0 1), for example, to make the system inconsistent.

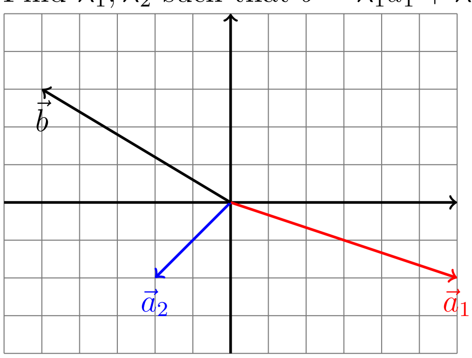
(4) If the augmented matrix of a system with four equations and four unknowns has five columns (one for each unknown and one for the constants), it is not possible to have a pivot in each column.

Question 7 Using the graphs below, find the coefficients of the requested linear combinations. There may be multiple solutions or no solution. In the graphs below, a square represents 1 unit.

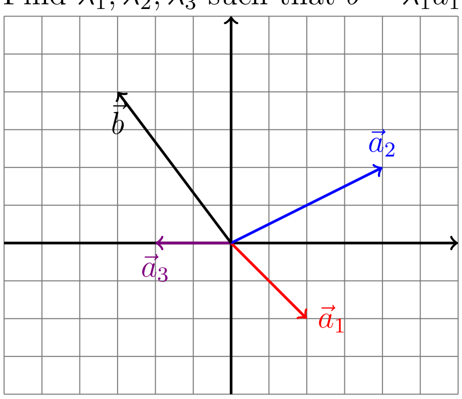
a) Find λ_1, λ_2 such that $\vec{b} = \lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2$



b) Find λ_1, λ_2 such that $\vec{b} = \lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2$



c) Find $\lambda_1, \lambda_2, \lambda_3$ such that $\vec{b} = \lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 + \lambda_3 \vec{a}_3$



Solution:

a) $\lambda_1 = -4, \lambda_2 = -2$

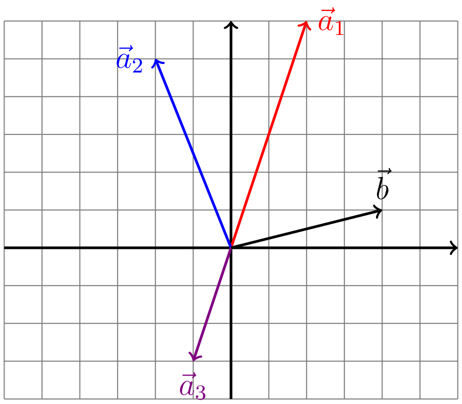
b) $\lambda_1 = -1, \lambda_2 = -1/2$

c) There is infinitely many solutions. Here is one : $\lambda_1 = -2$, $\lambda_2 = 0$ and

$\lambda_3 = -1/2$

Another method is to write the systems of equations corresponding to the vector equations that need to be solved.

Question 8 In the graph below, find $\lambda_1, \lambda_2, \lambda_3$ such that $\vec{b} = \lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \lambda_3\vec{a}_3$. Can μ_1 and μ_3 be found such that $\vec{b} = \mu_1\vec{a}_1 + \mu_3\vec{a}_3$?



Solution: There are infinitely many solutions. Here are two examples: $\lambda_1 = 0$, $\lambda_2 = -1$, and $\lambda_3 = -2$, or $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 0$. No, it is not possible to find μ_1 and μ_3 such that $\vec{b}$ is a linear combination of $\vec{a}_1$ and $\vec{a}_3$, because these two vectors are collinear.

Question 9 Let the vectors

$$\vec{a}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} 5 \\ -13 \\ -3 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} -3 \\ 8 \\ 1 \end{pmatrix}.$$

(1) Is it possible to write $\vec{b}$ as a linear combination of $\vec{a}_1$ and $\vec{a}_2$?

(2) Provide a geometric interpretation of the result.

Solution:

(1) Non. Considérons l'équation linéaire $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$, d'inconnues x_1, x_2. Le système linéaire correspondant est

$$\begin{cases} x_1 + 5x_2 &= -3 \\ -2x_1 - 13x_2 &= 8 \\ 3x_1 - 3x_2 &= 1 \end{cases}$$

with augmented matrix

$$\left(\begin{array}{rr|r} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{array}\right)$$

and RREF

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right).$$

The system has no solution.

(2) This means that the vector $\vec{b}$ does not belong to the plane formed by the vectors $x_1\vec{a}_1 + x_2\vec{a}_2$, with x_1 and x_2 real.

Question 10 Let

$$A = \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}$$

Show that the equation $A\overrightarrow{x} = \overrightarrow{b}$ is not consistent for every vector $\overrightarrow{b}$ in $\mathbb{R}^2$. Find and describe the set of vectors $\overrightarrow{b}$ for which $A\overrightarrow{x} = \overrightarrow{b}$ is consistent.

Solution: Since the second row of the matrix A is -2 times the first row, it is necessary and sufficient that the second component of $\overrightarrow{b}$ also satisfies this property: $b_2 = -2b_1$. Thus, for $\overrightarrow{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the system has no solution.