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# A General Outline of the Genesis of Vector Space Theory

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The following article presents a general outline of the genesis of the elementary concepts of vector space theory. It presents the main works that contributed to the development of these basic elements and analyzes how they developed and how they influenced each other. The study of systems of linear equations and the search for an intrinsic geometric analysis were the two main sources which gave rise to the theory of linearity. The fact of going beyond the third dimension in geometry in the middle of the 19th century, as well as the dialectical development between algebra and geometry from the creation of analytical geometry on, brought about the development of an initial unification of linear questions around the concept of determinant. This framework was generalized to the countably infinite dimension following work on functional analysis. Axiomatization, which was carried out at the end of the 19th century, although only really put to use after 1920, is a wider process which is part of the general development of mathematics in the beginning of the 20th century. I will analyze how this phenomenon came into existence and how it finally established its influence. © 1995 Academic Press, Inc.

L'article qui suit, présente une vue générale de la genèse des concepts élémentaires de la théorie des espaces vectoriels. Il présente les principaux travaux qui ont œuvré dans ce sens en analysant leurs interactions et les grandes lignes de développement. L'étude des systèmes d'équations linéaires et la recherche d'un calcul géométrique intrinsèque sont les deux principales sources de constitution d'une théorie de la linéarité. Le dépassement de la dimension 3 en géométrie au milieu du XIXème siècle, ainsi qu'un développement dialectique entre algèbre et géométrie depuis la création de la géométrie analytique ont amené la constitution d'une première unification des questions linéaires autour de la notion de déterminant. Ce cadre fut généralisé à la dimension infinie dénombrable lors de travaux d'analyse fonctionnelle. L'axiomatisation réalisée à la fin du XIXème siècle, mais vraiment utilisée seulement après 1920, est un processus plus large qui s'inscrit dans un développement général des mathématiques au début du XXème siècle. J'analyserai comment ce phénomène a pu naître et comment il a fini par s'imposer. © 1995 Academic Press, Inc.

El artículo que sigue presenta una visión general sobre la génesis de los conceptos elementales de la teoría de espacios vectoriales. Aqui se presentan los principales trabajos que se han desarrollado en este sentido, analizando sus interacciones y las grandes líneas de desarrollo. El estudio de sistemas de ecuación linear y la investigación de un cálculo geométrico intrínseco son las dos principales fuentes de constitución de una teoría de la línearidad. La superación de la dimensión 3 en geometría, a mediados del siglo XIX, asi como el desarrollo dialectico entre el algebra y la geometría, después de la creación de la geometría analítica, han llevado a la constitución de una primera unificación des los aspectos líneares alrededor de la notion de determinante. Este cuadro fue generalizado a la dimensión infinita enumerable con los trabajos de analisis funcional. La axiomatisación realizada a fines del siglo XX, pero realmente utilisada después de 1920, es un proceso más largo que se inscribe en un desarrollo general de la matemática a comienzos del siglo XX. Yo analizare como ese fenómeno pudo nacer y como él a terminado por imponerse. © 1995 Academic Press, Inc.

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## INTRODUCTION

The axiomatic theory of vector space is a recent achievement in mathematics. Giuseppe Peano gave the first axiomatic definition of a vector space in 1888 [76, 141–152], but the theory was not actually developed before 1920. From the thirties, this theory rapidly became a frame—almost a language—in use in many branches of mathematics as well as in various sciences. Not only did it serve as a basis for new discoveries, but it also helped to renew proofs or methods in already well-established areas of mathematics. In this sense, vector space theory is essentially a unifying and generalizing theory, as it gave more cohesion to mathematics as well as leading to new discoveries.

My goal in this paper is to recall the main aspects of the evolution of the concept of linearity up to its final stage in vector space theory. I focus my research on those elementary concepts, which appeared to be essential: linear combination, independence–dependence, rank–dimension, linear transformation [29]. I show how these different concepts have been developed in various fields of mathematics (although I concentrate on geometry and linear equations), and I try to point out the different stages in the process of unification and theorization of linear problems.

This work could have taken as its starting point ancient civilizations, all of which developed *ad hoc* techniques for solving linear equations. Nevertheless, until the middle of the 18th century, it can be said that, apart from the improvement of techniques to solve systems of linear equations and the development of symbolic algebra after François Viète and René Descartes, nothing substantial occurred with regard to linear algebra.

## LINEAR EQUATIONS: BUILDING THE FIRST CONCEPTS

The year 1750 is an important date for our subject, as it is the date of publication of two crucial works in the history of the vector space concept. The first is the famous treatise, *Introduction à l'analyse des courbes algébriques*, by Gabriel Cramer, in which he set up the frame for the theory of determinants [24].[1] The second is entitled *Sur une contradiction apparente dans la doctrine des lignes courbes* and was written by Leonhard Euler [34]. It concerns Cramer's paradox, which is related to algebraic curves. Two propositions were thought to be true, although only partially proven, at the beginning of the 18th century [34, 220–221]:

(1) Two distinct algebraic curves of order $m$ and $n$ have $mn$ points in common. It was known that some could be multiple, complex, or infinite, but mathematicians also knew of examples for which these points were all simple and real.

(2) $n(n + 3)/2$ points are necessary and sufficient to determine a curve of order $n$.

[1] The name "determinant" was introduced by Cauchy in [17]. In fact, in 1693, Gottfried Wilhelm Leibniz, in a letter to the Marquis de l'Hospital [67, 2:238–240] (for an English translation, see [88, 267–269]), had given the first notation with double index of a system of linear equations and a rule to calculate the determinant. This letter was only published for the first time in 1850, so Cramer's text is usually referred to as the starting point of the theory of determinants.

The paradox appears for $n$ more than two because then $n(n + 3)/2 \leq n^2$, so it seems that two algebraic curves may have more points in common than is sufficient to determine each of them. Colin MacLaurin, in 1720, was one of the first to identify this paradox, and Cramer reformulated it in 1750 in the treatise quoted above. Also in 1750, Euler identified the nature of the problem. In his work, after a close analysis of the situation, Euler explained that in some cases Proposition (2) might not be true, as $n$ equations might not be sufficient to determine $n$ values. He gave some examples to show that one equation might be comprised (*comprise* in French) in one or several others. This text is quite unusual for its time, as mathematicians then talked about linear equations only in order to develop methods for solving them. Euler's approach is more qualitative, and his text is one of the first in which the question of the dependence of linear equations is raised and pointed out, although the issue of the solution is still important, and marks a difference compared to the modern definition of linear dependence. Indeed, he starts with the case of two equations and gives the example "$3x - 2y = 5$ and $4y = 6x - 10$." He says "It is not possible to determine the two unknowns $x$ and $y$, as while eliminating $x$, the other disappears and an identical equation remains, of which nothing can be deduced. The reason for such an incident is at first quite obvious, as the second equation can be changed into $6x - 4y = 10$, which, being nothing but the double of the first $3x - 2y = 5$, does not differ from it at all" [34, 226].

No one can doubt that something so "obvious" could not have been noticed by any mathematician of Euler's time, but one equation being the double of the other is not sufficient in itself to assert that one unknown is not determined; Euler had to solve the system by elimination and substitution in order to give a convincing proof and to point out the actual "incident."

For three equations, he gave an example with two similar equations, and another example in which one equation is the double of the sum of the two others. In these two cases, there is no trial for solving the equations. The conclusion is: "Thus, when one says that to determine three unknowns, it is sufficient to have three equations, the restriction needs to be added that these three equations are so different that none is already comprised in the others" [34, 226].

For four equations, Euler remarked that, in some cases, two unknowns may not be determined, and he provided the following example:

$$5x + 7y - 4z + 3v - 24 = 0,$$

$$2x - 3y + 5z - 6v - 20 = 0,$$

$$x + 13y - 14z + 15v + 16 = 0,$$

$$3x + 10y - 9z + 9v - 4 = 0,$$

they are only worth two, as after extracting from the third the value of

$$x = -13y + 14z - 15v - 16,$$

and after its substitution in the second, one gets:

$$y = \frac{33z - 3v - 52}{29} \quad \text{and} \quad x = \frac{-23z + 33v + 212}{29};$$

> the substitution of these two values of $x$ and $y$ in the first and fourth equations leads to identical equations, therefore the quantities $z$ and $v$ will remain undetermined. [34, 227]

Here again the proof turns on eliminations and substitutions, and Euler does not mention any linear relations between the equations, although they are quite obvious ((1) − (2) = (4) and (1) − 2 × (2) = (3), for instance).

After these examples, he concluded with a general statement: "When one says that to determine $n$ unknown quantities, it is sufficient to have $n$ equations giving their mutual relations, the restriction must be added that they are all different or that none is confined [enfermée] in the others" [34, 228]. The terms "comprised" or "confined" are not clearly defined. In a modern context, they would point out a linear relation between the equations, but this is not exactly the meaning given by Euler. In his approach, these terms refer to an "incident" in the final process of elimination and substitution that results in one or several unknowns remaining undetermined. Of course, he notes, although not systematically, the linear relations between the equations, but his proofs never rely on this fact. So in the nature of this definition there is nothing *a priori* linear. This is why I propose to say that Euler introduced the notion of *inclusive dependence* rather than linear dependence. Of course, the two notions coincide when applied to linear equations, and the distinction may seem superfluous, but inclusive dependence is embedded in the context of equations and cannot be transferred into other linear situations (like $n$-tuples). We will see that this had an influence on the development of linear algebra over the next hundred years.

On the other hand, Euler's definition is very close to intuition and suited his purpose. When he treated the case $n = 4$ (cf. the quotation above), he developed arguments in which one can recognize an empirical intuition of the notion of rank. Moreover, at the end of his text, Euler developed similar considerations in relation to Cramer's paradox:

> When two lines of fourth order meet in 16 points, as 14 points, when they lead to different equations, are sufficient to determine one line of this order, these 16 points will always be such that three or more equations are already comprised in the others. In this way, these 16 points do not determine more than if there were 13 or 12 or even less points and in order to determine the curve entirely, one must add to these 16 points one or two others. . . . [34, 223]

This kind of reasoning on equations is based on intuition and uses little theory. Yet it was to remain an isolated approach. Indeed, after Cramer's work, the theory of determinants became a very prosperous branch of mathematical activity [74]. As a consequence, the study of linear equations became a part of this new theory and Euler's ideas did not immediately generate further investigations. In fact, for nearly a century, the questions related to undetermined and inconsistent systems of linear equations were neglected, whereas it is only through these questions that one can approach the notions of dependence and rank. From around 1840 to 1879, within the theory of determinants, the concepts of rank took shape. In the context of linear equations, rank is an invariant which determines the size of the set of solutions (minimal number of generators/maximal number of independent solutions) and, by a process of duality, the number of relations of dependence (minimal number of equations describing the set of solutions/maximal number of independent equations). To create the concept of rank containing these different aspects, mathematicians had to overcome several obstacles and change their point of view on certain elementary notions. I analyzed this process in detail in [30]. I give just the main outlines here.

One can list three main sources of obstacles and difficulties:

(1) the recognition of the invariance which was, if not unseen, at least assumed without necessity of proof,

(2) the possibility of the same definition of dependence between equations and $n$-tuples, and

(3) the anticipation of the concept of duality and the consideration of all the systems of equations which have the same set of solutions.

Of course, these three points are not independent, and the progress relative to each of them influenced the progress of the other two. The concept of inclusive dependence, as I pointed out in Euler's work, remained but was also rapidly connected to the evanescence of the main determinant of a square system of linear equations. Moreover, the notion of minor allowed a determination of the "size" of the set of solutions in relation to the maximal number of independent equations. Indeed, the maximal order $r$ of nonevanescent minors in a system of $p$ linear equations in $n$ unknowns gives the number $n$-$r$ of arbitrary unknowns to be chosen to describe the set of solutions of a consistent system and represents the maximal number of independent equations in the system. Such ideas became well known by the middle of the 19th century.[2] The classical method consists in first isolating the part of the equations corresponding to a nonzero minor of maximal order and then using Cramer's rule with the other unknowns as parameters appearing in the second members. This manipulation requires theoretical justifications, which prior discoveries on determinants made explicit. Moreover, this first phase opened new ways to more systematic investigation into systems of equations. However, it was more the search for a practical method than concern for theoretical achievement that led to this first step. Henry J. S. Smith's approach pointed out a change of point of view which marked a fundamental step. Indeed, in a paper of 1861 [89], he showed that the maximal order of a nonzero minor is also related to the maximal number of independent solutions. This does not help, directly, to describe the set of solutions better. In this sense it is more a theoretical than a practical result, and this shows the slight but decisive change of approach marked by Smith, who not only was interested in giving ways to solve the systems of equations, but also studied them on a theoretical basis.

Between 1840 and 1879 the concept of rank is, therefore, implicitly central to

[2] The first results in this direction were developed by several mathematicians (for a conjoint overview, I give some references in [74]: Sylvester in 1840 [91 or 74, 1:227–235], Cayley in 1843 [20 or 74, 2:14–17], Sylvester in 1850 [92 or 74, 2:50–52], Baltzer in 1857 [3], Trudi in 1862 [74, 3:84–85], Baltzer in 1864 [74, 3:227–235], Dodgson in 1867 [74, 2:85–86], Rouché in 1880 [85 or 74, 3:86–90], and Fontené [74, 3:90–92].

the description of systems of linear equations. With the use of determinants, an analogous treatment of the question of dependence relative to equations and $n$-tuples is possible. Yet the technicality of demonstrations involving determinants seems to have made it difficult to provide a clear and concise overview of all the relations of invariance and duality involved. The work developed in this period is very different from Euler's [34]. The tools are more sophisticated, but their use requires so much technique that intuition is averted.

Georg Ferdinand Frobenius managed to achieve this goal of clarity and conciseness, while also defining premises without the use of determinant. Indeed, in a paper of 1875, "Über das Pfaffsche Problem" [44], his first original idea was to give a common definition of independence for equations and $n$-tuples without using determinants:

> Several particular solutions
>
> $$A_1^{(\chi)}, \ldots, A_n^{(\chi)}, \quad (\chi = 1, \ldots, k),$$
>
> will be said to be independent or different, when $c_1 A_\alpha^{(\chi)} + \cdots + c_k A_\alpha^{(\chi)}$ cannot be zero for $\alpha = 1, \ldots, n$, without $c_1, \ldots, c_k$ being all zero, in other words, when the $k$ linear forms $A_1^{(k)} u_1 + \cdots + A_n^{(k)} u_n$ are independent. [44, 236]

This is the modern definition still in use today.

Frobenius then introduced the notion of "associate" ["zugeordnet" or "adjungirt"] system. A system of linear homogeneous equations being given, a homogeneous system will be called an "associate" system if the coefficients of its equations constitute a basis of solutions of the original system. Of course, this correspondence is a symmetrical relation. In the two first pages of this section, Frobenius set up the basis for the notion of duality by considering $n$-tuples and equations as similar objects that can be seen from two different angles. Given a system of $n$ linear homogeneous equations in $p$ unknowns, with maximal order of nonevanescent minor equal to $r$, he showed that one can find a maximum of $p$-$r$ independent solutions. Given one of this set of solutions (a basis), he built an "associate" system by simply reversing the role of coefficients and unknowns in the equations, and showed that any basis for solution of the "associate" system has an "associate" system with the same set of solutions as the initial system of equations. He then easily pointed out and related all the invariants attached to the number $r$. His approach used technical theorems and methods of the theory of determinants, but most of his results are expressed without the use of determinants, except for the potential definition of rank. In a paper of 1879, "Über homogene totale Differentialgleichungen" [45], he used the term rank for the first time and defined it in these terms: "When in a determinant, all minors of order $(m + 1)$ vanish, but those of order $m$ are not all zero, I call the *rank* [*Rang*] of the determinant the value of $m$" [45, 1].

With Frobenius, it can be considered that within the theory of determinants the concept of rank reached its maturity. This can be seen in a paper of 1905, entitled "Zur Theorie der linearen Gleichungen" [46], in which he gave a complete structured report of theoretical results on the study of linear equations.

A few interesting complementary results were established by Alfredo Capelli and Giovanni Gabrieri between 1886 and 1891 [74, 4:102–106]. First, they showed that any system of rank $r$ is equivalent to a triangular system of $r$ equations. Then they pointed out and proved that the rank of the lines of a matrix is the same as the rank of its rows. They also showed that a system of equations is consistent if and only if the rank of the array of their coefficients is the same as the rank of the array augmented by the row of second members.

At this stage of the analysis, the main aspect to keep in mind is that the study of linear equations and the theory of determinants represent the context in which the first theoretical concepts (dependence, rank, and duality) related to vector space theory were created and applied in finite dimension. Between 1750 and the beginning of the 20th century, determinants were omnipresent in all problems—both practical and theoretical—involving linearity (except in some works related to geometry, which will be studied in the following sections). This fact had an influence on the nature of the concepts, even if the role of determinants has been minimized by the axiomatic approach which gave a rather drastically different organization and order of priority in the coordination of elementary concepts. In particular, the concept of rank in the axiomatic theory of vector spaces is inseparable from the concept of dimension, which is a synthesis of the relations between the concepts of generators and dependence and can be (and usually is) introduced before the idea of duality. Still, even today, the concept of rank (not of dimension) is important in many problems of linear algebra, and the meaning it acquired over nearly two centuries, during which determinants were its support, is still an inevitable component of its nature.

## THE CONCEPT OF VECTOR IN GEOMETRY[3]

The relation between vector space theory and geometry seems somehow obvious to many people, because of the use of geometric representation to illustrate vectorial ideas, because of the use of a common vocabulary in the two fields, and because vectorial geometry is a very powerful method. The parallelogram of velocities, which is a geometric representation for the addition of vectors, has been used since the Middle Ages and even since Antiquity, but, as we shall see, it was far from adequate for the creation of the concept of directed line segment. Linearity refers to line in geometry, which is one of the basic figures. But the circle is also basic. The analytical method, introduced independently by René Descartes in his *Géométrie* [26] and Pierre de Fermat in *Ad Locos Planos et Solidos Isagoge* [37], organized geometry according to different criteria. The equation of a line being of the first order, it became the first level. The change of coordinates, useful, for instance, in the search for invariants to categorize curves, gave a framework for the study of linear transformations. Therefore, with the use of analytical methods in geometry, linearity became a starting point or a central question in many problems.

As early as 1679, in a letter to Christian Huygens (which remained unpublished

[3] For work complementary to the ideas presented in this section, see [25].

until 1833), Gottfried Wilhelm Leibniz criticized the analytical method and tried—although unsuccessfully–to create an intrinsic geometric analysis which he called the "geometry of situation" [66]. The reasons for his failure, so obvious to a modern reader, are less interesting than the content of his criticism, which gave an accurate analysis not only of the weakness of the analytical method but also of the nature of what an intrinsic geometric analysis should be. He wrote: "I am still not satisfied with algebra because it does not give the shortest methods or the most beautiful constructions in geometry. This is why I believe that, as far as geometry is concerned, we need still another analysis which is distinctly geometrical or linear and which will express situation directly as algebra expresses magnitudes" [67, 2:18–19]. And he continued: "Algebra is the characteristic for undetermined numbers or magnitudes only, but it does not express situation, angles and motion directly. Hence it is often difficult to analyze the properties of a figure by calculation, and still more difficult to find very convenient geometrical demonstrations and constructions, even when the algebraic calculation is completed" [67, 2:20].

The arbitrary choice of a system of coordinates being a transitory and external step in a geometrical demonstration was also a major philosophical concern in the use of the analytical method. From the beginning of the 19th century, the search for an intrinsic geometrical analysis became a preoccupation for several mathematicians on the basis of the criticism of the analytical methods as expressed above. In fact, an initial answer to this problem came, somewhat indirectly, from the geometrical representation of complex numbers. I say indirectly, because the principles for the geometrical representation of complex numbers came not only from the search for a geometric analysis but also, and sometimes mainly, from the motivation to legitimize the use of these numbers which were rejected by mathematicians as being inadequate to mathematical reality.[4] The question of the geometrical representation of complex numbers has been analyzed in various historical works (e.g., [15; 60, 3:117–158; 18; 11; 25; or 2]). In John Wallis's work [96, 2:286–295], one finds an initial attempt to illustrate complex numbers in geometry, but his model of gained and lost surfaces under the sea failed to give an illustration of multiplication. Within a few years, and independently, five practically unknown mathematicians from various countries set up the principle of the geometric representation of complex numbers: Caspar Wessel in 1799 [99], l'abbé Buée in 1805 [12], Jean Robert Argand in 1806 [1], C. V. Mourey in 1828 [73], and John Warren in 1828 [97]. However, it was only with Carl Friedrich Gauss around 1831 [49] and Augustin-Louis Cauchy around 1849 [18], that these principles became widely known and accepted among mathematicians. Complex numbers provided a model for a bidimensional geometrical analysis. In some of the works quoted above (especially Wessel's), the authors tried to generalize their ideas on space geometry, but their attempts always foundered on the difficult problem of multiplication. During the same period of time, two mathematicians—August Ferdinand Möbius and Giusto Bellavitis—developed

[4] They were still called impossible or imaginary quantities.

two different systems of geometrical analysis valid for dimension three as well as for dimension two, which laid the basis for vectorial geometry.

Möbius was one of the first mathematicians to have drawn out the notion of directed line segment. In the first chapter of his *Barycentrische Calcul* [70], the principles of which were first conceived around 1818, he designated a line segment from a point A to a point B by the notation AB and stated that AB = −BA [70, 1–5]; he then defined the addition of collinear segments. The central theorem of barycentric calculus is:

> Given any number ($\nu$) of points, A, B, C, ..., N with coefficients $a, b, c, \ldots, n$ where the sum of the coefficients does not equal zero, there can always be found one (and only one) point S—the centroid—which point has the property that if one draws parallel lines (pointing in any direction) through the given points and the point S, and if these lines intersect some plane in the points A′, B′, C′, ..., N′, S′, then one always has:
>
> $$a.\mathrm{AA}' + b.\mathrm{BB}' + c.\mathrm{CC}' + \cdots + n.\mathrm{NN}' = (a + b + c + \cdots + n).\mathrm{SS}'$$
>
> and consequently if the plane goes through S itself, then
>
> $$a.\mathrm{AA}' + b.\mathrm{BB}' + c.\mathrm{CC}' + \cdots + n.\mathrm{NN}' = 0.$$
>
> [70, 9–101]

In the case where the sum of the coefficients equals zero, Möbius noted, without further comment, that the point is sent to infinity.[5] Möbius' theory provided an algebra of points, but his goal was not to present an algebraic "structure" in all its details; he wanted rather to exhibit a tool for solving geometrical and physical problems. Indeed, the applications he gave are convincing and numerous. Moreover, his influence is important on a theoretical level; he inspired Christian von Staudt in the invention of projective coordinates, which freed projective geometry from any metric consideration and allowed a better understanding of the nature of projective properties.

It is somehow surprising that Möbius did not define the addition of noncollinear segments in his barycentric calculus; in fact, he did so eventually, but only in 1843, in his *Elemente der Mechanik des Himmels* [71]:[6]

> The position of a point B towards a point A is given, when its distance from A and the direction of the line AB are given. A third point B′ will therefore have the same position towards a fourth point A′, as B towards A, if the segments AB and A′B′ have the same length and direction (not opposite), which will be shortly formulated by: AB ≡ A′B′. With this characterization, we can easily prove the following theorems, with elementary geometry:

[5] In his 1844 *Ausdehnungslehre* (cf. below), Grassmann rediscovered barycentric calculus as an application of his more general theory, and he studied this case, showing that the linear combination of points is to be considered as a vector, when the sum of the coefficients is zero. Grassmann had great admiration for Möbius, who was one of the rare mathematicians to recognize his value; they wrote to each other over a long period.

[6] Möbius may have been influenced by Bellavitis, who had already discovered his *Calcolo delle Equipollenze* (see below) and had written to Möbius in 1835 [72, 4:717–718]. He had already been in contact with Grassmann as early as 1840.

I. If AB ≡ A′B′, then we also have: AA′ ≡ BB′.
II. If AA′ ≡ BB′ and BB′ ≡ CC′, then we also have: AA′ ≡ CC′.
III. If AB ≡ A′B′ and BC ≡ B′C′, then we also have: AC ≡ A′C′. [71, 1–2]

He also defined the multiplication by a non-negative number.

Finally, in 1862, Möbius wrote "Über geometrische Addition und Multiplication," which was only published in 1887, in the first edition of his collected work after being revised in 1865 [72, 4:659–697]. There, he defined addition of noncollinear segments, multiplication by any number, and two kinds of products of segments, which were directly inspired by Grassmann's work.

Möbius had the recognition of various famous mathematicians including Gauss, Cauchy, Jacobi, and Dirichlet (cf. Baltzer's remarks in [72, 1:xi–xii]). He created an efficient and practical method of solving geometrical problems; but although he pointed out some fundamental aspects of vectorial geometry, his theory, based on an intuitive perception of space, failed to offer the possibility of extension towards a more general concept of vector (or barycentric) space.

With his *Calcolo delle Equipollenze,* Giusto Bellavitis may be considered as the first mathematician to have defined, in 1833 [7], the addition of vectors in space:

(2°) Two straight lines are called *equipollent* if they are equal, parallel and directed in the same sense.
(3°) If two or more straight lines are related in such a way that the second extremity of each line coincides with the first extremity of the following, then the line which together with these forms a polygon (regular or irregular), and which is drawn from the first extremity of the last line is called the *equipollent-sum.* [8, 246]

He also defined the multiplication of coplanar directed line segments. In fact, the calculus of equipollences offered no more possibilities than complex numbers. Bellavitis himself admitted that his discovery was based on his reading of Buée's work [12], but throughout his life, he refused to accept complex numbers as part of mathematics. Indeed, his presentation is especially original for two main reasons: the objects on which the calculus is created are purely geometrical entities (not like complex numbers), and the first part of the calculus can be applied in space geometry, although, like many others, Bellavitis failed to generalize the product of directed line segments to space. This generalization was to be achieved by the Irish mathematician, Sir William Rowan Hamilton.

Hamilton had long been interested in a generalization to three dimensions of the geometric representation of complex numbers, when he finally invented the quaternions around 1843. His philosophical position towards the nature of algebra and his "Science of Pure Time" have been discussed in several works (for references, see [61] and [25, 17–46]). For the purpose of this work and in this paragraph, I will focus on the role of quaternions in the evolution of the concept of vector. From 1835 at least, Hamilton sought the equivalent of complex numbers for dimension three. Like all his predecessors, and quite naturally, he was looking for triplets with an addition and a multiplication. He had established the list of properties the two operations should have (equivalent to the structure of a field). After several attempts, all of which failed, Hamilton changed his point of view slightly by focusing on the geometrical nature of multiplication in dimension 2 rather than on general algebraic properties. He then pointed out that this multiplication is based on the ratio of lengths of the two vectors and the angle they form. He then transposed this idea in dimension three. An analysis of the problem [59, 1:106–110] led him to show that the multiplication in dimension 3 should take into account the ratio of lengths, and the rotation between the two directions of the vectors. The first is a one-dimensional value, and the second depends on the direction of the axis of the rotation (a two-dimensional value) and the angle (a one-dimensional value).

This change towards the geometric signification of multiplication pointed out the fact that quadruplets suited the problem of three-dimensional geometric algebra, whereas triplets did not. It also showed the impossibility of preserving commutativity, as rotations in space do not commute. In 1844, Hamilton was able to publish the first results of his discovery [58]. The quaternions are algebraic numbers which allow a geometric representation in space. The multiplication represents, at the same time, the scalar product and the vector product. Hamilton's discovery had widespread influence on the development of vector analysis [25, 17–46]. Yet this development, initiated by quaternions, is not the most important for the theory of vector spaces. Indeed, it is mostly because of the change they introduced in algebra that quaternions had a strong influence on the emergence of linear algebra. This point will be discussed below.

More generally, the elaboration of systems of geometric calculus seems to have had, in the long term, more influence on the development of vector analysis than on the theory of vector space. Yet its historical importance should not be underestimated. Indeed, geometry is a central part of mathematics, potentially rich in questioning. The possibility of a geometric interpretation of algebraic results is therefore a source of enrichment as it gives to concepts an intuitive background and more consistency. The use of geometric terms in the general theory of vector spaces is proof of this fact and highlights the privileged relations between geometry and linear algebra.

## LINEAR ALGEBRA IN THE DIALECTIC EXCHANGES BETWEEN GEOMETRY AND ALGEBRA

The attempts described above to create an intrinsic geometrical analysis can be viewed either as a desire to free geometry from the external invasion of arithmetic or as an attempt to import some aspects of algebra into geometry. In any case, from the discovery of the analytical method, the new relation between algebra and geometry meant that the evolution of the two fields was henceforth intrinsically linked in a dialectical process. In this sense, the use of the analytical method in geometry generated the creation of most of the tools of matrix algebra through the study of linear substitutions (i.e., change of coordinates). The concept of geometric transformation as something which applies to the whole space or plane is recent and was really only fully established through Felix Klein's *Erlangen Program* [64]. Yet many problems in analytical geometry led to the application of changes of coordinates, and the study of linear substitutions (as they were known at the time) is similar to that of linear transformations. Moreover, the use of linear substitution appeared not only in geometry but also, for instance, in the study of quadratic forms with integral coefficients in various problems of arithmetic and in the solution of differential equations.

In 1770, Euler, in "Problema algebricum ob affectiones prorsus singulares memorabile," studied questions which can be interpreted in terms of orthogonal linear substitutions [35]. In fact, the objects he was interested in were squares of numbers, which he compared to a magic square:

$$\begin{matrix} A, B, C, \\ D, E, F, \\ G, H, I, \end{matrix}$$

which satisfies the following conditions:

$$\begin{aligned} &1.\ A^2 + B^2 + C^2 = 1, && 4.\ AB + DE + GH = 0, \\ &2.\ D^2 + E^2 + F^2 = 1, && 5.\ AC + DF + GI = 0, \\ &3.\ G^2 + H^2 + I^2 = 1, && 6.\ BC + EF + HI = 0, \\ &7.\ A^2 + D^2 + G^2 = 1, && 10.\ AD + BE + CF = 0, \\ &8.\ B^2 + E^2 + H^2 = 1, && 11.\ AG + BH + CI = 0, \\ &9.\ C^2 + F^2 + I^2 = 1, && 12.\ DG + EH + FI = 0. \end{aligned}$$

[35, 75]

He first established that these 12 conditions are equivalent to the fact that the transformation

$$X = Ax + By + Cz;\ Y = Dx + Ey + Fz;\ Z = Gx + Hy + Iz;$$

is such that $XX + YY + ZZ = xx + yy + zz$ [35, 77]. He also noted that the first 6 relations imply the last 6. Then he showed that these relations are equivalent to 9 others, which, in modern terms, states that the matrix equals the opposite of the matrix of cofactors.[7] He did not raise the question of independence of the relations, but used the intuitive reasoning that the $n^2$ coefficients being bound by $n(n + 1)/2$ conditions, an orthogonal substitution depends on $n(n - 1)/2$ parameters.

For $n = 3$ he showed that an orthogonal transformation can be written as the product of $n(n - 1)/2 = 3$ orthogonal substitutions which only modify two axes at a time. Euler's method introduced three angles (which are still known in mechanics as Euler's angles) to characterize an orthogonal substitution. Moreover, Euler did not stop with $n = 3$; using only algebra, he gave a full solution for $n = 4$ and 5. Finally, he said that for any $n$, an orthogonal substitution can be represented by $n(n - 1)/2$ parameters as its $n^2$ coefficients depend on $n(n + 1)/2$ relations.

Euler's approach remained essentially algebraic (even if his results have very interesting geometrical interpretations for $n = 3$), which is why he did not limit

[7] In fact, Euler made a small mistake that led him to ignore positive isometries. Indeed, if $A$ is an orthogonal matrix, then $A^{-1} = A^t$ and $A^{-1} = (1/\det(A))C(A)^t$, therefore $A = (1/\det(A))C(A)$. Thus, Euler's proof is only valid when $\det(A) = -1$.

himself to dimension three; the situation was quite different in geometry, as we will see below.

Between 1773 and 1775 in his "Recherche d'arithmétique," Joseph Louis Lagrange was led to study the effect of linear substitutions with integral coefficients in a quadratic form of two variables, while studying the properties of numbers which are the sum of two squares [65, 3:695/795]. He established the fact that the discriminant (he did not give it a name) of the new quadratic form is the product of the old discriminant by the square of a quantity which has since been known as the determinant of the linear substitution. Around 1798, in his *Disquisitiones arithmeticae,* Gauss studied the same question with two and three variables [48]. He introduced a notation which was very similar to a matrix to characterize the linear substitution and, moreover, established the formula for the composition of two linear substitutions:[8]

> 268. If a ternary form $f$, in the variables $x, x', x''$, and determinant $D$, is transformed into a ternary form $g$, whose determinant is $E$, by the substitution
>
> $$x = \alpha y + \beta y' + \gamma y'',\quad x' = \alpha' y + \beta' y' + \gamma' y'',\quad x'' = \alpha'' y + \beta'' y' + \gamma'' y'',$$
>
> the coefficients $\alpha, \beta, \gamma, \alpha', \ldots$ being integers, we will say briefly that the form $f$ is transformed into $g$ by the substitution
>
> $$\alpha, \beta, \gamma;\ \alpha', \beta', \gamma';\ \alpha'', \beta''', \gamma'' \ldots \text{(S)},$$
>
> and that $f$ contains $g$ or that $g$ is contained in $f$. . . .
>
> 270. If the ternary form $f$ contains the ternary form $f'$ and this one contains $f''$, the form $f$ also contains $f''$. Thus it is easy to see that if $f$ is transformed into $f'$ by the substitution
>
> $$\alpha, \beta, \gamma;\ \alpha', \beta', \gamma';\ \alpha'', \beta''', \gamma'',$$
>
> and $f'$ into $f''$ by the substitution
>
> $$\delta, \varepsilon, \zeta;\ \delta', \varepsilon', \zeta';\ \delta'', \varepsilon'', \zeta'',$$
>
> $f$ will be transformed into $f''$ by the substitution
>
> $$\begin{array}{lll} \alpha\delta + \beta\delta' + \gamma\delta'', & \alpha\varepsilon + \beta\varepsilon' + \gamma\varepsilon'', & \alpha\zeta + \beta\zeta' + \gamma\zeta''; \\ \alpha'\delta + \beta'\delta' + \gamma'\delta'', & \alpha'\varepsilon + \beta'\varepsilon' + \gamma'\varepsilon'', & \alpha'\zeta + \beta'\zeta' + \gamma'\zeta'', \\ \alpha''\delta + \beta''\delta' + \gamma''\delta'', & \alpha''\varepsilon + \beta''\varepsilon' + \gamma''\varepsilon'', & \alpha''\xi + \beta''\xi' + \gamma''\xi''; \ldots \end{array}$$
>
> [48, 306–309]

This marks a fundamental step towards the concept of a matrix; not only did this text introduce a symbolic notation for linear substitutions, but it used it to represent multiplication.

Yet, for a certain time, the concept of determinant was not clearly identified as separate from the concept of matrix, which was a source of confusion. For instance, this might have been a reason why the noncommutativity of the product of matrices had not been pointed out whereas, on the contrary, it was known that det($AB$) =

[8] In his memoir on determinants [17], Cauchy showed that the product of two determinants of the same order is also a determinant, and he recognized the influence of Gauss in his discovery.

det($A$)det($B$) = det($BA$). There are, however, some even more fundamental reasons, like the fact that the multiplication of matrices was not seen as an algebraic operation but just as a local process. This attitude towards the algebraic status of matrices changed around the middle of the 19th century, when matrices appeared to be connected to various objects and when recent developments (like the discovery of quaternions) had enlarged the field of algebra. From this perspective, the English algebraic school—most notably Arthur Cayley and James Joseph Sylvester—was the most active center for progress in this direction. But one can find similar ideas around the same time in Germany, for instance. Indeed, one of the first attempts to give a systematic listing of algebraic properties of matrices appeared in Ferdinand G. M. Eisenstein's work [33]; in particular, noncommutativity is singled out. He also used one single letter to refer to a matrix and to describe algebraic relations, which was one of the most important changes in the treatment of matrix operations. The study of the algebraic operations on matrices (square or rectangular) reached a first stage of maturation with the publication in 1858 of Cayley's famous "Memoir on the Theory of Matrices" [22], in which the author gathered, in a detailed and carefully organized report, all the results discovered within the two preceding decades.

As I pointed out above, the use of coordinates made the generalization of geometry to more than three dimensions possible and, in a way, natural. But the desire for legitimacy towards reality prevented such a generalization until the middle of the 19th century. For instance, Möbius, in his barycentric calculus, defined two figures as equal and similar when they are such that every point of the first figure can be associated with a point in the other, so that the distance between any two points in the first figure is equal to the distance between the two associated points in the second figure [70, 181–183]. He then showed that two equal and similar figures in the same plane can be brought into coincidence, and he remarked that this is not true for solid figures. He concluded that

> for the coincidence of two equal and similar systems A, B, C, D, ... and A′, B′, C′, D′, ... in space of three dimensions, in which the points D, E, ... and D′, E′, ... lie on opposite sides of the planes ABC and A′B′C′, it will be necessary, we must conclude from analogy, that we should be able to let one system make a half revolution in a space of four dimensions. But since such a space cannot be thought of, so is also coincidence, in this case, impossible. [70, 526]

Möbius' attitude is usual for his time. He could have ventured into space of more than three dimensions, but, since it lay beyond the realm of possibility, neither he nor his contemporaries went further into this field. Cayley, in 1846, was one of the first to have made a decisive step in this direction. In his paper, entitled "Sur quelques résultats de géométrie de position," he showed how one can obtain results in three-dimensional geometry by working in a space of more than three dimensions:

> One can, *without using any metaphysical notions towards the possibility of a four-dimensional space,* reason as follows (everything may also be easily translated into purely analytical language): Assuming four dimensions in space, one must consider *lines* determined by two points, *half-planes* determined by three points, *planes* determined by four points; (the intersection of two planes is therefore a half-plane, etc.). Ordinary space must be considered as a plane, and

its intersection with another plane is an ordinary plane, with a half-plane, an ordinary line, and with a line, an ordinary point. [21, 217–218]

On the other hand, the development of mathematics in the 19th century facilitated and justified the use of spaces of more than three dimensions. In this sense, two types of events were fundamental:

(1) The discussions of the foundations of geometry, with the discovery of non-Euclidean geometry and the development of projective and algebraic geometry, enlarged the traditional field of investigation of geometry.

(2) The discovery of the quaternions by Hamilton annihilated George Peacock's principle of equivalent forms, which stated that any algebra should have the law of arithmetic as a foundation. This opened the way to many discoveries of new types of algebra and ultimately resulted in the evolution of algebra as an independent field of arithmetic. (See [75; 80; 78; 79].)

In the second half of the 19th century, linear algebra still did not exist as a unified field, but a geometry in dimension $n$ was developed on the basis of analytic geometry and the theory of determinants and matrices. In this new field, one of the most productive branches was the study of $n$-ary forms which was the continuation of the work of Lagrange and Gauss on quadratic forms. Many mathematicians worked in this direction, and their investigations led, among other things, to most of the results concerning the reduction of matrices.

## GRASSMANN'S *AUSDEHNUNGSLEHRE:* AN ISOLATED SINGULARITY[9]

In 1844, Hermann Grassmann published the first version of his *Lineale Ausdehnungslehre* (literally "linear theory of extension") [51]. This was announced by the author as the first part of a general theory, *die Ausdehnungslehre,* never completed by Grassmann. This work was highly original for its time and remains so today. The roots of the *Ausdehnungslehre* are varied from both a mathematical and a philosophical point of view, but geometry and the nature of space represent important sources of Grassmann's reflection, even though his theory of extension contains more ambitious prospects. Grassmann, who studied theology and philosophy, was a self-taught mathematician and was mainly influenced by the work of his father, Justus Grassmann. His research on geometrical analysis can be traced back at least to 1832; in 1840, in order to obtain a better position as a secondary school teacher, he wrote a long essay (over 200 pages) on the theory of tides ("Theorie der Ebbe und Flut"), which was published only posthumously in his collected works [53, 3:1–238]. In this work, he laid the bases of his geometrical analysis and gave substantially simplified solutions to many results in Lagrange's *Mécanique analytique* and Laplace's *Mécanique céleste.* In the foreword of the *Ausdehnungslehre,* he recalled the origins of his mathematical inspiration:

[9] For more details on Grassmann, one can consult, e.g., [68; 69; 38; 39; 40; 41, preface; 31; 32].

> The initial incentive was provided by the consideration of negatives in geometry; I was used to regarding displacements AB and BA as opposite magnitudes. From this it follows that if A, B, C are points of a straight line, then AB + BC = AC is always true, whether AB and BC are directed similarly or *oppositely,* that is even if C lies between A and B. In the latter case AB and BC are not interpreted merely as length, but rather their directions are simultaneously retained as well, according to which they are precisely oppositely oriented. Thus the distinction was drawn between the sum of lengths and the sum of such displacements in which the directions were taken into account. From this there followed the demand to establish this latter concept of a sum, not only for the case that the displacements were similarly or oppositely directed, but also for all other cases. This can most easily be accomplished if the law AB + BC = AC is imposed even when A, B, C do not lie on a single straight line.
>
> Thus the first step was taken toward an analysis that subsequently led to the new branch of mathematics presented here. However I did not then recognize the rich and fruitful domain I had reached; rather, that result seemed scarcely worthy of note until it was combined with a related idea.
>
> While I was pursuing the concept of product in geometry as it has been established by my father,[10] I concluded that not only rectangles but also parallelograms in general may be regarded as products of an adjacent pair of their sides, provided one again interprets the product, not as the product of their lengths, but as that of the two displacements with their directions taken into account. When I combined this concept of the product with that previously established for the sum, the most striking harmony resulted....
>
> This harmony did indeed enable me to perceive that a completely new domain had thus been disclosed, one that could lead to important results....
>
> Thus I felt entitled to hope that in this new analysis I have found the only natural way in which mathematics should be applied to nature, and likewise that in which geometry should be treated if it is to lead to general and fruitful results. [51, v–viii]

Grassmann claimed that he had created a new theory which, although it could be applied to geometry, mechanics, and various other scientific fields, was independent of them. Indeed, he thought that geometry should not be part of mathematics, since it refers to reality which validates some of its processes: it is a science outside mathematics, and the theory of extension is the mathematical model to be applied to it. Moreover, Grassmann made his theory self contained and independent of the rest of mathematics, in the sense that it relied only on the elementary rules of mathematical reasoning. As a consequence, it included many preliminary definitions and introduced many new notions with new words, which Grassmann carefully took from German rather than Latin roots. Grassmann also had precise ideas on the type of presentation mathematical work should follow: "Now we characterize a method of treatment as scientific if the reader is thereby on the one hand led necessarily to recognize the individual truths, and on the other is placed in a position from which he can survey the broader sweep of the development" [51, xxix–xxx].

This implies a dialectical presentation based on the contrast between formal and real aspects of the theory: general results must be deduced from general objects and their properties, but at each step of the theory, the reader must be able to see from which concrete situation and by which means the general theory proceeds. The choice of presentation is essential in Grassmann's work; this was also an obstacle

[10] Cf. J. G. Grassmann, *Raumlehre,* Teil II, p. 194, and *Trigonometrie,* p. 10 (Berlin; G. Reimer, 1824 and 1835). [This note is Grassmann's.]

that most of Grassmann's contemporaries did not get beyond. As a matter of fact, critics of Grassmann's work often pointed out its lack of clarity, due to an overwhelming tendency to mix up mathematical results with obscure philosophical considerations. Grassmann was also often reproached for giving applications only after general results, which made his ideas very hard to follow.

For instance, Ernst Friedrich Apelt, in a letter to Möbius, written on 3 September, 1845, asked:

> Have you read Grassmann's strange *Ausdehnungslehre?* I know it only from Grunter's *Archiv;* it seems to me that a false philosophy of mathematics lies at its foundation. The essential character of mathematical knowledge, its intuitiveness [Anschaulichkeit], seems to have been expelled from the work. Such an abstract theory of extension as he seeks could only be developed from concepts. But the source of mathematical knowledge lies not in concepts but in intuition. [53, 3:101; English translation, 25, 79]

Ernst Eduard Kummer, who was asked to give an expert report on Grassmann's work wrote in a similar vein on 12 June, 1847:

> Regarding first what concerns the form or the representation of the treatise, one has to admit in general that it is a failure; for, even though the style is good and full of spirit, it lacks everywhere a suitable organization of its content in which essential points could be clearly distinguished from things of less importance. [41, 19]

Grassmann's philosophical position was not common in his time and cannot be separated from his mathematical work, of which it is an essential component. In his thoroughly documented paper [68], Albert C. Lewis has pointed out many aspects of this fact, and, in particular, the influence of Schleiermacher's *Dialektik* on Grassmann, the main result of which is the use of a dialectic of contrast as an important source of progress in Grassmann's theory. The contrasts used by Grassmann can be listed pairwise: equal–different, discrete–continuous, general–particular, real–formal, etc. Grassmann's mathematical work could not and still cannot be fully appreciated, if one does not make the effort to understand the philosophy on which it is based. In 1862, Grassmann published a completely revised version of the *Ausdehnungslehre* [52], from which most of the philosophical considerations had been deleted, and he also adopted a more classical mathematical presentation. Nevertheless, this version met with no more success than the first one. Certainly, the disappearance of any philosophical background, if it avoided an initial superficial obstacle, made the mathematical content somewhat difficult to accept. Moreover, as in 1844, many readers were discouraged by the strict Euclidean organization, which did not permit a partial reading of the theory, as one had to read it from the first page in order to understand the meaning of any concept.

In spite of this lack of success, Grassmann's theory contained the bases for a unified theory of linearity, as it introduced, with great accuracy and in a very general context, elementary concepts such as linear dependence, basis, and dimension. Furthermore, the mathematical content went quite beyond this point, and some of the concepts introduced by Grassmann have been a source of inspiration for recent theories, such as Elie Cartan's exterior algebra [16] or more recently, Gian-Carlo Rota's exterior calculus [84].[11] Let us now turn to a brief analysis of some of Grassmann's key ideas.

The 1844 *Ausdehnungslehre* starts with a long philosophical introduction, in which Grassmann explains what mathematics is for him, compared to philosophy or other sciences. This is followed by an introductory chapter about the "General Theory of Forms," which attempts to formalize the concept of algebraic addition, subtraction, multiplication, and division. This could be seen as quite close to an axiomatic presentation of the structures of group, ring, and field, but, as Lewis noticed, "what is presented is not a set of unproven statements from which succeeding statements are deduced; rather, principles of connection, expressed by means of the general concepts equality and difference, and connection and separation, are symbolized" [68, 140].

This introductory chapter gives the rule for the investigation of the formal aspect of the theory of extension as well as the rules for the construction and comparison of new entities by connections with others. Generation is an important concept in Grassmann's work. Entities are not given *a priori* and are not defined according to the properties of their operations; they are created through the "evolution" or the connection of other entities.

The concept of "extensive magnitude" is introduced as follows: a given element is to generate a "system of first order" by the "continuous action of the same fundamental evolution" (or its opposite); then another "evolution," applied to each element of the system of first order will generate a system of second order, etc., with no limitation on the number of orders. The concept of "evolution" corresponds in geometry to a movement along a straight line, but in Grassmann's theory, it has a more general meaning, based on "the fundamental intuition of space and time," which is given "*a priori,*" and is "originally inherent to us like the body is to the soul" (cf. foreword of [51]). Furthermore, Grassmann, careful to develop an autonomous theory, did not use the concept of number at the beginning of his theory. He deduced it, in the fourth chapter only, from the concept of division of colinear extensive magnitudes. This choice restricted him from using any multiplication by a scalar (until Chapter 4) and therefore any linear combination. This made the first third of the 1844 version rather difficult to understand (at least for a modern reader). Still this framework, with its intuitive basis, proved its effectiveness in generating a rich model for linearity, as Grassmann managed to define the essential objects and to prove most of the elementary properties of finite-dimensional vector spaces.

His approach to the concepts of bases and dimension is particularly interesting. According to the original mode of generation (which represents the real aspect of the theory), a system of $n$th order is generated by $n$ fundamental methods of evolution, which are given as independent (i.e., none is included in a system gener-

[11] See also Arno Zaddach's study of Grassmann's *Algebra* [101].

ated by some of the others). Therefore, the order of a system, which is the "natural" dimension, is intrinsically related to the concepts of generation and dependence; it represents the measure of the extension. In order to contrast this particular mode of generation by a general approach to the theory, the main explicit goal of the first seven paragraphs of Grassmann's theory (Section 13 to 20) is to make the higher order system independent from this initial model of generation. This is done by using the contrast between the formal and real aspects of the addition of displacement (the equivalents of our vectors) [51, Section 17–19], in order to attain the final result: "[A] system of $m$th order is generable by any $m$ methods of evolution belonging to it that are mutually independent" [51, 30].

This provides a notion equivalent to the modern concept of basis and gives the value $m$ a general meaning close to the concept of dimension. Nevertheless, one aspect is missing, as nothing proves that fewer than $m$ methods of evolution could not generate the system. In fact, in his proof Grassmann gives a result from which this fact can easily be deduced:

> First I will show that if the system is generated by $m$ methods of evolution whatever, I can replace any given one of them by a new method of evolution ($p$) belonging to the same system of $m$th order and independent of the remaining ($m$-1), and, using this in combination with the other ($m$-1), generate the given system. [51, 30]

This is exactly what is now known as the exchange theorem. In 1844, Grassmann did not explicitly deduce from this result that a system of $m$th order cannot be generated by less than $m$ methods of evolutions, although he admitted it implicitly. In 1862, the exchange theorem and its consequences were given explicitly in a series of six theorems [52, 19–21]. Yet, as we shall see below, most of Grassmann's readers did not see the importance of this result.

Linear dependence and dimension are also central concepts in Grassmann's theory, as they are in modern vector space theory. In the two versions, he gave an elegant proof of a result equivalent to the formula about the dimension of the sum and intersection of two subspaces [51, 183–185]:

$$\dim(E + F) + \dim(E \cap F) = \dim E + \dim F.$$

The two concepts are related to the two types of products (exterior and regressive) which are original creations of Grassmann, but a detailed analysis of these would take us too far afield in this paper.

As mentioned above, Grassmann employed a very different theoretical framework in 1862. In particular, objects are given *a priori* and defined through operations, which is closer to modern presentations: given a system of $m$ units (i.e., $m$ linear independent magnitudes), Grassmann defined a system of order $m$ as the system of all linear combinations of the units. Then he defined addition and subtraction, and multiplication and division by a number. A list of "fundamental properties" for the four operations followed, with the acknowledgment that all the algebraic laws of addition, subtraction, multiplication, and division will follow from them.

8. For extensive magnitudes $a$, $b$, $c$, the following fundamental formulas hold:

(1) $a + b = b + a$

(2) $a + (b + c) = (a + b) + c$

(3) $a + b - b = a$

(4) $a - b + b = a. \ldots$

12. For the multiplication and division of extensive magnitudes ($a$, $b$) by numbers ($\alpha$, $\beta$) the following fundamental formulas hold:

(1) $a\beta = \beta a$

(2) $a\beta\gamma = a(\beta\gamma)$

(3) $(a + b)\gamma = a\gamma + b\gamma$

(4) $a(\beta + \gamma) = a\beta + a\gamma$

(5) $a.1 = a$

(6) $a\beta = 0$ if and only if $a = 0$ or $\beta = 0$

(7) $a{:}\beta = a(1/\beta)$ if $\beta \neq 0$. [52, 15–16]

This presentation can be seen as a kind of *a posteriori* axiomatization of linear structure. Moreover, the fundamental properties given by Grassmann are almost the same as the axioms of the modern structure of vector space, except for (1) and (7) about multiplication, which are mere conventions; (6), which is a redundant property; and the ambiguous use of subtraction, which made the concept of zero and opposite somewhat unclear. In fact, this last point had been carefully analyzed in the general theory of forms of the 1844 version [51, 8–9]. This shows that the 1862 version could not be read independently of the original 1844 work. Moreover, the mathematical content was not drastically changed in 1862, in spite of some improvements and new applications.

In many ways, Grassmann's theory remains a singularity. Even if all its results correspond to modern concepts and theories, it contributed to the creation of very few of them. For the theory of vector spaces, it played an important role in the discovery of the axiomatic theory, but most of the concepts of this theory were reestablished independently of Grassmann's work. Yet, because of its original approach, the 1844 version remains unique and offers an alternative to vector space theory, which gives a rich analysis of the relations between the formal concepts of linear algebra and geometrical intuition that cannot be found elsewhere.

## THE FIRST AXIOMATIC APPROACHES

In 1888, Peano published a condensed version of his own reading of Grassmann's *Ausdehnungslehre* entitled *Calcolo geometrico.* At the end of this treatise, in a small final chapter, he gave an axiomatic definition of what he called a *linear system,* which is the first axiomatic definition of a vector space (in modern terms) [76, 141–142]. Peano's axioms are very similar to Grassmann's fundamental properties, but Peano's contribution is fundamental; he really put forward the properties of the operations to describe the structure, unlike Grassmann who had deduced them from the definition of the operations on the coordinates. Moreover, Peano, aware of axiomatic approaches, improved the formulation of properties by suppressing the convention and the redundancy built into Grassmann's fundamental properties; he also defined more clearly the concepts of zero and opposite elements. Yet this definition was not quickly followed by many new developments, and his approach was not taken up immediately.

Following Peano, in 1897, Cesare Burali-Forti published a book on differential geometry [13] in which he used Grassmann's ideas and pointed out Peano's role in making the *Ausdehnungslehere* more accessible. Nothing is said, however, about the axiomatic approach introduced in the *Calcolo geometrico.* In 1909, with another Italian mathematician, Roberto Marcolongo, Burali-Forti published *Omografie vettorialli con applicazioni alle derivate respetto ad un punto et alla fisica-matematica* [14], which opened with an axiomatic presentation of linear systems. However, the definition Burali-Forti and Marcolongo gave conforms less than Peano's to the model of an axiomatic definition (some axioms are redundant and some are missing), and they did not provide more substantial results and applications. But their work is important because it is the first to have such an introduction (in Peano's book it appeared only in the final chapter). Although their work does not seem to have generated immediate followers either, it helped make the axiomatic approach known (if not used) especially in Italy and in France.

In spite of their important contribution to axiomatic vector space theory, one has to admit that Peano, Burali-Forti, and Marcolongo were, on some points, less accurate than Grassmann. This is especially true relative to the concept of dimension. Indeed, they defined it, before the concept of basis, as the maximal number of independent vectors in the space. This is sufficient to prove easily that any set of $n$ independent vectors in an $n$-dimensional space constitutes a system of generators and therefore a basis. They overlooked however the question of minimality of a system of generators, which is necessary for the proof of the uniqueness of the number of elements in a basis. Nor did they quote the exchange theorem, which is closely related to the problem of minimality of generators. In fact, they focused on applications to geometry, using the axiomatic definition as mere formality. What they failed to retain from Grassmann was the richness of the dialectic between the real and formal aspects which governed his process of generalization. Therefore, dimension, for Peano or Burali-Forti and Marcolongo, was, as in geometric space, a measure of the degree of freedom, whereas for Grassmann, it was a measure of extension. Indeed, space is more often characterized by its limitation to three dimensions than by its expansion from one point into three dimensions.

In the first edition of his *Raum–Zeit–Materie* of 1918 [100], Hermann Weyl also gave an axiomatic definition of what he called linear vector-manifolds. He did not mention his Italian predecessors, but he referred to Grassmann's *Ausdehnungslehre* as an "epoch making work" [100, 17, note 4]. Indeed, his definition is closer to

Grassmann's 1862 fundamental formulas than to Peano's definition. Regarding the question of dimension, Weyl chose to add a *dimensional axiom* to his definition: "There are $n$ linearly independent vectors, but every $n + 1$ are dependent on one another" [100, 17]. But he also commented that: "From this property (which may be deduced from our original definition with the help of elementary results on linear equations) it follows that $n$, the dimensional number, is as such a characteristic of the manifold, and is not dependent on the special vector base by which we map it out" [100, 17].

So Weyl's approach to the concept of dimension did not differ that much from Peano's, except that his note shows concern about the invariants related to the number of dimensions. Moreover, Weyl seemed to refer to the fact that if all vectors of a linear vector-manifold E derive from $n$ vectors, then any set of $n + 1$ are dependent on each other.[12] This result is another preliminary theorem (different from the exchange theorem) from which the invariance of the number of elements in a basis may be proven. Yet, this is still ambiguous, even if Weyl's approach is considered as an improvement over that of the Italians.

In fact, in 1862, Grassmann had made a remark about the central theorem on dimension that he had deduced from the exchange theorem: "This important theorem can also be derived directly from the theory of elimination.... But the proof presented here is not only elementary, but in addition has the advantage that thereby the essential, simple relation between the extensive magnitudes appears more clearly" [52, 21]. This proves that Grassmann was far ahead of his time on several levels.

Moreover, the question of dimension was discussed, somehow more accurately, in the theory of fields. Indeed, a field extension is a vector space on the original field whose dimension is the order of extension. In this context, as in Grassmann's *Ausdehnungslehre,* the notion of generation was intrinsically present in the first notions of the theory. Although the question of the invariance of the number of elements in a basis of the field extension was considered obvious for a while, it was raised and solved with great care by Richard Dedekind in 1893 in the 11th supplement to the fourth edition of Gustav Peter Lejeune-Dirichlet's *Vorlesungen über Zahlentheorie* [28]. In this supplement, one paragraph constitutes a very general approach to linear structure, and gives it a form very modern for its time.

Dedekind opened with the definition and properties of linear independence (he called it irreducibility). Then he defined a space [Schaar] $\Omega$ as the set of all linear combinations of an irreducible set of $n$ numbers in a field $A$. He called these $n$ elements a basis of $\Omega$ and defined the coordinates of an element of $\Omega$. Immediately thereafter, he presented three properties that he proved to be characteristic of $\Omega$:

[12] This is an obvious consequence of the fact (well known since around 1800) that a system of $n + 1$ homogeneous linear equations in $n$ unknowns always has a nonzero solution.

I. The numbers of $\Omega$ reproduce themselves through addition and subtraction, i.e., the sum and the difference by any two numbers are in $\Omega$.

II. Any product of a number in $\Omega$ by a number in $A$ is a number of $\Omega$.

III. There are $n$ independent numbers in $\Omega$, but any $n + 1$ such numbers are dependent. [28, 468]

To deduce the properties from the definition, as Dedekind pointed out, only the second part of III required a proof, and for that he used induction. Assuming the property for any space having a basis of less than $n$ elements, he took $n + 1$ elements $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n$ in a space $\Omega$ with a basis of $n$ elements.

If one is zero, for instance $\alpha = 0$, then these elements are dependent; if not, one can assume that for instance the first coordinate of $\alpha$ is not zero; so one can obviously find $n$ numbers $c_1, c_2, \ldots, c_n$, in $A$, such that the first coordinate of each number

$$\alpha_1 + c_1\alpha, \ \alpha_2 + c_2\alpha, \ldots, \alpha_n + c_n\alpha,$$

is zero; thus these numbers belong to a space whose basis contains only the $n - 1$ elements $\omega_2, \omega_3, \ldots, \omega_n$, and are therefore dependent on each other; consequently there are $n$ numbers $a_1, a_2, \ldots, a_n$, in $A$, which are not all zero and such that

$$a_1(\alpha_1 + c_1\alpha) + a_2(\alpha_2 + c_2\alpha) + \cdots + a_n(\alpha_n + c_n\alpha) = 0,$$

and as the sum $a = a_1c_1 + a_2c_2 + \cdots + a_nc_n$ is also in $A$, it follows that the $n + 1$ numbers $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n$, are dependent on each other. [28, 468]

This result is equivalent to what Grassmann obtained with the exchange theorem, although the approach is quite different. Moreover, this proof does not use the theory of linear equations although it does use representation with coordinates.

The proof that these three properties are characteristic of a space as defined by Dedekind was quite simple; he also immediately deduced that any irreducible system of $n$ numbers is a basis of $\Omega$. Raising the problem of change of basis, he showed moreover that a system of $n$ numbers is irreducible if and only if the determinant of their coordinates on the original basis is not zero. The impossibility of finding a system of less than $n$ generators is not explicitly stated, but as with Grassmann in 1844, this question could be easily solved, since the definition put the emphasis on the original set of $n$ independent generators.

At the end of the paragraph, Dedekind turned to the case of an extension field, i.e., when the space is also closed under multiplication. In this context, Dedekind proved the following main results:

(1) In an extension field with a basis of $n$ elements, any number is algebraic and its grade is at most $n$.

(2) Such an extension field $B$ is said to be of $n$th grade with respect to $A$, and therefore any system of $n$ independent elements constitutes a basis of $B$.

(3) If $\theta$ is algebraic with respect to $A$ and of grade $n$, then $A(\theta)$ is an extension field of $A$ of $n$th grade and $1, \theta, \theta^2, \ldots, \theta^{n-1}$ constitute a basis of $A(\theta)$.

In this last supplement, Dedekind also studied the properties of modules, and especially of their bases, but this goes beyond the scope of the present study. In Section 164, the main results of which have just been sketched, Dedekind's approach was very close to a modern axiomatic presentation of the elementary results on finite dimensional vector spaces. The question of dimension was examined with care, and using original tools, compared to Grassmann's use of the exchange theorem and what had been done in the theory of equations.

In 1910, Ernst Steinitz published his *Algebraische Theorie der Körper* [90], which marked an important stage in the history of modern algebra and which served as a reference work for at least a quarter of a century. In this major work, Steinitz gave a precise definition of linear dependence over a field R, and defined a finite extension of order $n$: "Let R be a subfield of L, L will be said to be finite with respect to R and of order $n$—which will be designed by: [L:R] = $n$—if there are in L, $n$ elements linearly independent over R, while any set of more than $n$ elements of L are linearly dependent over R" [90, 198]. This definition is identical to that given by Peano or Burali-Forti and Marcolongo for the number of dimensions of a linear space. Since the Italians' works were practically unknown in Germany, however, it likely had no influence on Steinitz's or Weyl's formulation. Explicit references to Dedekind are also absent in the works of Weyl and Steinitz, but there is a natural proximity which has to be taken into account.

After this definition, Steinitz showed that in an extension of order $n$, for any set of $n$ elements, the linear independence is equivalent to the fact that any element of the extension cannot be expressed in more than one way as a linear combination of these $n$ elements. This follows quite obviously from the preceding definition, and indeed there is nothing new compared to his Italian predecessors. In fact, Steinitz pursued a goal that he had not explicitly stated. For him, a basis of L is a set of elements such that any element of L can be expressed uniquely as a linear combination of them. Steinitz aimed to prove that any basis has $n$ elements and that any set of $n$ independent elements is a basis of L. To achieve this goal, he needed to prove that the order of an extension cannot exceed the number of generators. In other words, he had to connect a result concerning generation with a property of dependence. He cast his definition of a basis in terms of the coordinate system, so his proof was set up within the context of $n$-tuples and linear equations:

> If one knows that every element $\beta$ of L can be expressed as: $\beta = c_1\alpha_1 + \cdots + c_n\alpha_n$; then for any $(n + 1)$ elements $\beta_0, \beta_1, \ldots, \beta_n$, there are expressions like:
>
> $$\beta_i = c_{i1}\alpha_1 + \cdots + c_{in}\alpha_n; \qquad (i = 0, \ldots, n)$$
>
> and therefore, according to the theorem of Section 1 about homogeneous linear equations,[13] $(n + 1)$ elements $d_0, d_1, \ldots, d_n$ can be found such that $d_0\beta_0 + d_1\beta_1 + \cdots + d_n\beta_n = 0$. The extension L is thus finite and its order is at most $n$. [90, 200]

After Grassmann's 1862 *Ausdehnungslehre*, this was the first explicit proof of the fact that a set of generators cannot have fewer elements than the number of dimensions, although this was implicit in Dedekind's work and very easily deducible.

[13] This theorem proves that a system of less than $n$ homogeneous linear equations with $n$ unknowns has a nonzero solution.

Yet the approaches and tools of Steinitz and Grassmann were very different. Grassmann knew that a proof could be given with the use of elimination theory, but he preferred the exchange theorem for explicit reasons given in 1862 (cf. his remark quoted above).

In fact, further on in his theory, Steinitz used something very close to the exchange theorem, but in the context of transcendental extensions where algebraic dependence replaces linear dependence: "Let S be a system of elements of an extension field L, an element *a* of L is said to be algebraically dependent on S (with respect to R), if *a* is algebraic with respect to the field R(S)" [90, 288]. Two systems are said to be equivalent if any element of each system is algebraically dependent on the other system. On the other hand, a system is said to be irreducible if it is not equivalent to any of its subsystems. Transferred into the context of linearity, two equivalent irreducible systems correspond to two bases of the same linear space. After a few preparatory results, Steinitz gave three final theorems which contain, for algebraic dependence, results equivalent to the exchange theorem, the theorem about the completion of an independent system into a basis, the invariance of the number of elements in the bases of a linear space, and properties of the dimension of a subspace of a finite dimensional linear space:

> 7. Let S be an irreducible system (with respect to R), *a* a transcendental element with respect to R, but algebraically dependent on S, then S contains a finite subsystem L with the following properties: *a* is algebraically dependent on L; any subsystem of S, on which *a* is algebraically dependent, contains the system L; if any element of L is replaced by *a*, S is changed into an equivalent irreducible system; none of the other elements of S have the same property....
>
> 8. Let U and B be finite irreducible systems with respectively $m$ and $n$ elements; let us assume that $n \leq m$ and B is algebraically dependent on U. Then if $m = n$ the systems U and B are equivalent, and if on the contrary $n < m$, U is equivalent to an irreducible system made of B and $m - n$ elements of U....
>
> 9. An irreducible system B, which is algebraically dependent on a finite system U, cannot have more elements than this one; in case they have the same number of elements they are equivalent. [90, 290–292]

Steinitz presented these results in such a progressive, deductive way that they are close to an axiomatic definition of a general concept of dependence (which would include algebraic as well as linear dependence) from which the concepts of dimension and basis could be deduced. Therefore, little needs to be done in order to translate the preceding results into the linear context, but Steinitz did not mention (at least here) this possibility. About twenty years later, two of his successors, Bartel L. van der Waerden in [95, 1:96] and Emanuel Sperner setting up Otto Schreier's work in [87, 20], used the exchange theorem (and used this term) for one of the first times, in the context of linear dependence. Although both of them referred to Steinitz, I have never found proof that Steinitz actually used the exchange theorem in the context of linear dependence. On the other hand, there is no reference to Grassmann, although his work was well known by German algebraists of this time. Nevertheless, even if an influential filiation could be conjectured, in spite of the gap of nearly a century, the two approaches are embedded in very different mathematical and philosophical contexts.

## DIFFERENTIAL EQUATIONS AND FUNCTIONAL ANALYSIS

Differential equations have been an important point of interest in mathematics since the 18th century, and their study became the starting point of what would be dubbed functional analysis around the turn of this century.[14] Furthermore, the study of linear differential equations played an important part in the theorization of linearity.

In the middle of the 18th century, Jean le Rond D'Alembert, Lagrange, and Euler had noticed that the general solution of a linear homogeneous differential equation of order $n$ could be expressed as a linear combination of a set of $n$ "fundamental" solutions. This was not expressed in these terms, of course, and no rigorous proof was given. Moreover, the question of the independence of the "fundamental" solutions had not been raised. In fact, the solution was considered as a power series in the neighborhood of each point, and its derivatives were obtained by taking the derivative of each term of the series. Then, the summation, term by term, led to a linear recurrent equation which determined the coefficients of the series as functions of the $n$ first, according to a well-established method (see, e.g., [65, 4:151–165]). Around the same time, it was fairly commonly known that the general solution of a differential equation could be obtained by the addition of a particular solution and the general solution of the homogeneous equation, and by around 1770, Lagrange had discovered his method of "variation of constants" to find a particular solution for the whole equation [65, 4:159]. Nevertheless, it was only in the first half of the 19th century that Cauchy, in his *Cours de l'Ecole Polytechnique,* clarified these notions and gave rigorous proofs.

Moreover, the developments in the study of differential and partial differential equations led to sophisticated questions dealing with linearity. In this framework, in many occurrences, one can trace unlabeled and more or less implicit methods or concepts which are today essential components of linear algebra. For instance, as early as 1770, Lagrange introduced a method using what would today be called an adjoint operator; he also used, implicitly, some tools of duality to solve systems of linear differential equations [65, 1:472–478]. The spectral theory and the concepts of eigenvalue and eigenvector were developed to a great extent in the field of differential equations. For instance, twice around 1762 and 1776, Lagrange, while solving systems of simultaneous differential equations with several functions, used similar methods which involved the search for what we now call the eigenvalues of a matrix, but he did not mention the similarity of the two problems [65, 1:520–534 and 6:655–666]. Indeed, most of the concepts and methods remained implicit and ununified for at least another century.

[14] For more details on this subject, see [27].

In 1822, in his *Théorie analytique de la chaleur,* Joseph Fourier, while solving differential equations by power series, used methods to solve systems of countably infinite linear equations in countably infinite unknowns [42, 168 or 212]. The lack of understanding of the convergence of power series did not allow him to give a correct solution to his problem, but this work was to give the basic principle of the method: study the square finite subsystem of order *n* and then let *n* tend to infinity. In fact, for more than half a century, no one really investigated the solution of infinite linear systems. One of the first texts presenting consistent results on this type of method was published in 1886 by Henri Poincaré who referred to a text by George William Hill of 1877 [77]. After Poincaré's work, many mathematicians such as, David Hilbert [63], Frédéric Riesz [81], Erhard Schmidt [86], Jacques Hadamard [54], and Maurice Fréchet [43], studied infinite systems of linear equations. Their methods were based on Fourier's ideas, but they considered different restrictive boundary conditions on the power series which ensured the convergence of the infinite determinant. Of course, the difficult goal of their works was mainly to obtain a result with the most general boundary conditions. In fact, most of what was known in the finite dimensional case was investigated in the context of countably infinite dimension. Therefore, up to the 1920s and, in many ways, even in the following decade, the theories of determinant, matrix and quadratic and bilinear forms, generalized to countably infinite dimension served as the framework for a unified theory of linearity.

Yet some mathematicians started to feel the inaccuracy of some methods which had become highly technical and difficult to manipulate. For instance in a text of 1909, Otto Toeplitz proved certain theorems without the use of determinants, but with a sophisticated use of elimination, to obtain triangular equivalent systems [94]. Moreover, gradually, mathematicians changed their approach to the problem by considering more and more general vector spaces of functions; this would lead finally to the axiomatization of functional analysis. One of the most decisive steps in this direction was made by Riesz, in a paper of 1916, which was translated into German in 1918 under the title, "Über lineare Funktionalgleichungen" [82]. This is one of the first papers to give a general definition of a normal and of a closed vector subspace of functions:

> We call [the set of continuous functions from [a, b] into R] to be brief, the *functional space.* Moreover we call *Norm* of f(x) the maximum value of $|f(x)|$, and we denote it by $\|f\|$; the magnitude $\|f\|$ is therefore always positive and is zero only if f(x) is always zero. The following relations hold:
>
> $$\|cf\| = |c|\,\|f\|;\ \|f_1 + f_2\| \leq \|f_1\| + \|f_2\|\ldots.$$
>
> We come now to the similar problem for *linear transformations.* A transformation T, which associates to every element f of our functional space another element T(f), will be said to be linear, when it is *distributive* and *bounded.* The transformation is said to be distributive if the following properties hold for any *f*:
>
> $$T[cf] = cT[f],\ T[f_1 + f_2] = T[f_1] + T[f_2].$$

And T is said to be bounded when there is a constant M, such that, for all *f*:

$$\|T(f)\| \leq M\|f\|.$$

[82, 72]

Riesz's main achievement was to establish the foundation of what is now known as the Riesz–Fredholm theory of compact operators. Although the entire paper is set in the space of continuous functions on a real compact interval, most of the results could be generalized to other functional spaces, as, most of the time, only the axiomatic definitions are used. In 1921, Eduard Helly considered a general normed sequence vector space and hence also marked a new stage in the axiomatization of functional analysis [62]. Finally, the decisive step towards axiomatization was taken independently by Stefan Banach in his thesis, defended in 1920 and published in 1922 [6, 2:306–348], and Hans Hahn, in two papers of 1922 [55] and 1927 [56].

In the introduction to his dissertation, entitled *Sur les opérations dans les ensembles abstraits et leut applicatinoux equations integrables* Banach stated that:

> The present book follows the goal of establishing a few theorems valid for various functional fields, which I will specify. Nevertheless, so that I do not have to prove them separately for each field, which would be painful, I have chosen a different method, that is: I will consider in a general sense the sets of elements of which I will postulate certain properties, I will deduce some theorems and then I will prove for each specific functional field that the chosen postulates are true. [6, 2:308]

This quotation shows that Banach was fully aware of what an axiomatic approach was, and of what he could do with it, and, indeed, axiomatic approaches were quite familiar among Polish mathematicians at that time. The basic structure defined in his work, as in Hahn's, is what we call today a Banach space (i.e., a complete normed vector space). Banach makes no reference to Peano or Burali-Forti or anyone else, but his definition of a vector space is closer to that used by Burali-Forti and is also not strictly correct as some axioms are redundant and some are missing. The functional spaces he has to deal with are mostly of uncountably infinite dimension, which makes the use of an axiomatic approach compulsory. In 1932, he published a treatise, entitled *Théorie des opérateurs linéaires* [5], in which he gave the general framework and most of the results of axiomatic functional analysis and infinite-dimensional linear algebra; this book was an enormous success and rapidly opened a new era in these two fields of mathematics.

## CONCLUSION

Many examples of implicit linear methods in different contexts in mathematics or physics can be traced from Antiquity, but, until the 19th century at least, they remained isolated methods; the same author could use the same idea twice (in terms of the theory of linear algebra) in different contexts without noticing the similarity of the methods. In this sense, linear algebra long remained implicit because of a lack of unification. Nevertheless, its origins can be found in various contexts, and linearity has always been a question which has penetrated nearly every branch of mathematics.

The first step on the way to a unified theory was made through the use of determinants. Until the beginning of the 20th century, this was the only point of commonality in the treatment of many linear problems. Here, I have analyzed how the elementary concepts of a theory of linearity were set up within this context. Furthermore, in the middle of the 19th century, the generalization of geometrical questions to spaces of more than three dimensions reinforced the cohesion of linear problems. The model of $R^n$, with the use of tools and concepts inherited from the theory of determinants and the use of geometric vocabulary and visual representations, gradually formed the theoretical background for any question of linearity. This process, which unified questions of linearity both inside and outside geometry, was very important for the further development of vector space theory. The above analysis underscores the differences in the nature of the contributions from geometry and from the theory of determinants to the formation of a theory of vector spaces and shows how that theory resulted from a synthesis of the two.

The 19th century also witnessed a fundamental change in the nature of algebra in general which contributed to the development of a consistent algebraic framework for the theory of linearity, with the investigation of the operations on $n$-tuples and on matrices. In this process, the use of geometric language provided an intuitive background for the theory. On the other hand, as noted, the study of systems of linear equations was generalized to countably infinite dimension by the end of the 19th century. In spite of the importation of tools from analysis like the notion of the convergence of series, this generalization corresponded to a linear process; it improved the foundation of the previous unification and extended the field of prospect, but it did not radically change the main concepts and tools on which it was based.

In the meantime, the concept of algebraic structure was emerging from a need to unify a rapidly expanding number of new theories in mathematics. Dedekind gave the first definitions of an axiomatic type of ring, ideal, field, and module in 1893 [28]. In his *Lehrbuch der Algebra* of 1894, Heinrich Weber gave the first axiomatic definition of a group [98]. The first thirty years of the 20th century, found several mathematicians, especially in Germany, setting up the framework for what would soon be called modern algebra. The theories of groups and field extensions brought out the most important issues in this new approach. In 1930, van der Waerden published the first volume of the first edition of his *Moderne Algebra;* the second volume was published the following year [95]. In this first edition, linear algebra centered on the structure of module, although most of the first concepts and results such as linear combinations, dependence, basis, and dimension were, in fact, established in a previous chapter on field extensions. But in the later editions, the place of linear algebra and vector space became more important and central. The study of systems of linear equations was then presented as an application of the theory of vector spaces and the role of determinants was considerably reduced. Rapidly, the popularity of modern algebra changed the way problems dealing with linearity were solved: determinants were dropped for the axiomatic approach which unified finite and infinite-dimensional questions. In 1941, Garrett Birkhoff and Saunders MacLane published their *Survey of Modern Algebra* [9], and in 1942, Paul R. Halmos's *Finite-Dimensional Spaces* [57] appeared. These two books were among the first to attempt to present the new theories for educational purposes to undergraduate students. In 1947, in France, Nicolas Bourbaki published the second chapter of book II of his *Eléments de mathématique* under the title *Algèbre linéaire* [10]. At first, this publication remained in the shadow of Halmos's very popular book, but its influence became stronger as Bourbaki's fame increased. Nevertheless, these three books have had a notable and long influence on the axiomatic theory of vector spaces, both in its use in mathematics and in its teaching.

The quasi-simultaneous publication of the first edition of van der Waerden's *Moderne Algebra* and Banach's *Théorie des opérateurs linéaires* marked two major events in the history of modern mathematics, which were to be essential in the unification of an axiomatic theory of vector spaces of finite or infinite dimension. From that time on, this theory has increasingly moved towards the central position it now holds in mathematics, as a basis for more elaborate theories and as a general framework for modeling many problems within and outside of mathematics.

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