Lecture Notes
Analysis II

For Engineering Students

Spring Semester 2025

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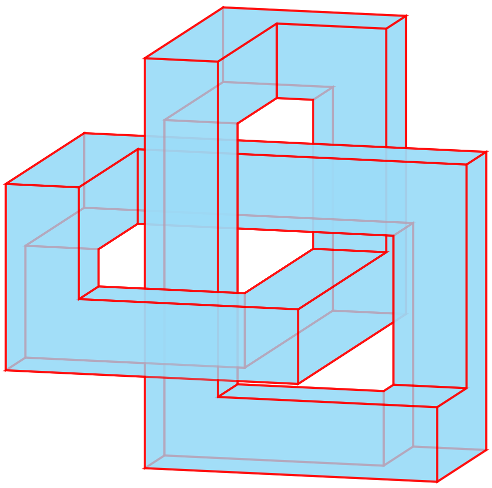
Chapter 1

The Euclidean space $\mathbb{R}^n$

In Analysis 1 you have learned the fundamental concepts of differential and integral calculus of real-valued functions in one real variable, known as *Single Variable Calculus*. However, real-life phenomena often depend on a multitude of factors and it requires more than just one variable to properly model such situations. This leads to the study of the theory of differentiation and integration of functions in several variables, called *Multivariable Calculus*. The mathematical stage on which the study of functions in several variables unfolds is the n-dimensional Euclidean space $\mathbb{R}^n$.



Before defining the n-dimensional Euclidean space and its intrinsic topology, let us recall some basic notions commonly used in analysis and calculus.

- $\mathbb{N}$ the *natural numbers* $\{1, 2, 3, 4, \ldots\}$,
- $\mathbb{Z}$ the *integers*, i.e., signed whole numbers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$,
- $\mathbb{Q}$ the *rational numbers* $\frac{a}{b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$,
- $\mathbb{R}$ the *real numbers*,
- $\mathbb{C}$ the *complex numbers*,

An *open interval* is an interval that does not include its boundary points and is denoted by parentheses. The open intervals are thus one of the forms

$$\begin{aligned}
(a, b) &= \{x \in \mathbb{R} : a < x < b\}, \\
(-\infty, b) &= \{x \in \mathbb{R} : x < b\}, \\
(a, +\infty) &= \{x \in \mathbb{R} : a < x\}, \\
(-\infty, +\infty) &= \mathbb{R},
\end{aligned}$$

where a and b are real numbers with $a \leqslant b$. The interval $(a, a) = \emptyset$ is the empty set, a degenerate interval. Open intervals are *open sets* in the topology of $\mathbb{R}$.

A *closed interval* is an interval that includes all its boundary points and is denoted by square brackets. Closed intervals take the form

$$\begin{aligned}
[a, b] &= \{x \in \mathbb{R} : a \leqslant x \leqslant b\}, \\
(-\infty, b] &= \{x \in \mathbb{R} : x \leqslant b\}, \\
[a, +\infty) &= \{x \in \mathbb{R} : a \leqslant x\}, \\
(-\infty, +\infty) &= \mathbb{R},
\end{aligned}$$

Closed intervals are *closed sets* in the topology of $\mathbb{R}$. Note that the interval $\mathbb{R} = (-\infty, +\infty)$ is both open and closed at the same time.

A *half-open interval* is a finite interval that includes one endpoint but not the other. It can be left-open or right-open, depending on which endpoint is excluded:

$$\begin{aligned}
(a, b] &= \{x \in \mathbb{R} : a < x \leqslant b\}, \\
[a, b) &= \{x \in \mathbb{R} : a \leqslant x < b\},
\end{aligned}$$

Note that half-open intervals are neither open nor closed sets in the topology of $\mathbb{R}$.

Intervals of the form $[a, b]$, $[a, b)$, $(a, b]$, (a, b) for $a, b \in \mathbb{R}$ with $a \leqslant b$ are called *bounded intervals*, whereas intervals like $(-\infty, b]$, $(-\infty, b)$, $[a, +\infty)$, and $(a, +\infty)$ are *unbounded intervals*.

1.1 The vector space $\mathbb{R}^n$

Given a positive integer n, the set $\mathbb{R}^n$ is defined as the set of all ordered n-tuples $(x_1, \ldots, x_n)$ of real numbers. It is called the *standard Euclidean space of dimension* n, or simply the n*-dimensional Euclidean space*.

We can represent an element of $\mathbb{R}^n$ either as an n-tuple, which is the same as a row vector with n entries,

$$\mathbf{x} = (x_1, \ldots, x_n)$$

or as a column vector with n entries

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

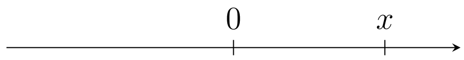
Both representations are common and widely used in the literature. We will generally use column vectors to denote elements of $\mathbb{R}^n$ in calculations, and row vectors to denote elements of $\mathbb{R}^n$ as input parameters of functions defined on $\mathbb{R}^n$.

There are also different ways in which elements in $\mathbb{R}^n$ are denoted, the three most common are

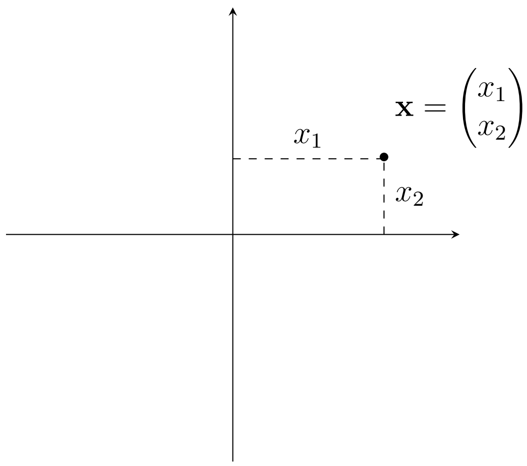
$$x, \qquad \mathbf{x}, \qquad \text{and} \qquad \vec{x}.$$

In this text, we will predominantly use x for elements in $\mathbb{R}$ and $\mathbf{x}$ for elements in $\mathbb{R}^n$ for $n \geqslant 2$.

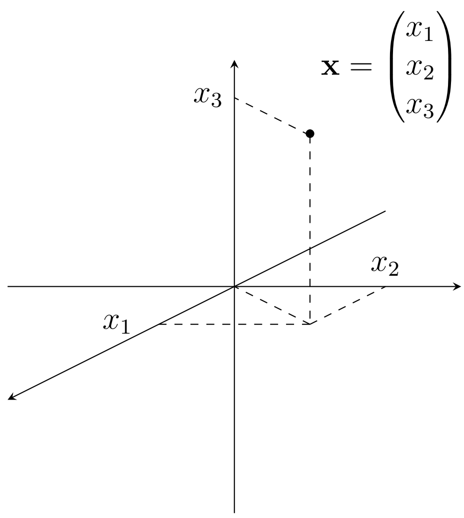
If $n = 1$ then $\mathbb{R}^1 = \mathbb{R}$ corresponds to the real line.



If $n = 2$ then $\mathbb{R}^2$ corresponds to the 2-dimensional plane. A point in $\mathbb{R}^2$ is usually written as either (x, y) or $\mathbf{x} = (x_1, x_2)^\top$.



If $n = 3$ then $\mathbb{R}^3$ corresponds to the 3-dimensional space. A point in $\mathbb{R}^3$ is usually written as eitehr (x, y, z) or $\mathbf{x} = (x_1, x_2, x_3)^\top$.



The set $\mathbb{R}^n$ is an n-dimensional inner product vector space over the real numbers. This means it is closed under addition, scalar multiplication, and endowed with an inner product called the scalar product. The addition on $\mathbb{R}^n$ is defined coordinate wise by

$$\mathbf{x}+\mathbf{y} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}.$$

The multiplication of an element $\mathbf{x} \in \mathbb{R}^n$ by a scalar $\lambda \in \mathbb{R}$ is defined as

$$\lambda\mathbf{x} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}.$$

The way in which addition and multiplication on $\mathbb{R}^n$ interact is described by the distributive law, which asserts that

$$\lambda(\mathbf{x}+\mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}. \tag{Distributive Law}$$

The vector space $\mathbb{R}^n$ is also equipped with a *scalar product* $\langle .,. \rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} x_k y_k. \tag{1.1}$$

Properties of the scalar product: The scalar product satisfies the three following properties:

1. **Positive-definiteness**: $\langle \mathbf{x}, \mathbf{x} \rangle \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$, with equality only for $\mathbf{x} = \mathbf{0}$.
2. **Symmetry**: $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
3. **Bilinearity**: $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z} \rangle = \alpha\langle \mathbf{x}, \mathbf{z} \rangle + \beta\langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$.

In linear algebra, a vector $\mathbf{x}$ is also an $n \times 1$ matrix. Its transpose, written $\mathbf{x}^\top = (x_1, \dots, x_n)$, is therefore a $1 \times n$ matrix, and we can interpret the scalar product of two vectors $\mathbf{x}, \mathbf{y}$ as the matrix product of $\mathbf{x}^\top$ and $\mathbf{y}$:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \cdot \mathbf{y} = (x_1, \dots, x_n) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

1.2 The Euclidean distance on $\mathbb{R}^n$

To be able to extend the analytical methods presented in Analysis 1 to the space $\mathbb{R}^n$, it is important to endow $\mathbb{R}^n$ with a topological structure. On $\mathbb{R}$ we have used the absolute value to define a distance $d(x,y) = |x-y|$, which was then used to define notions such as convergence and continuity in $\mathbb{R}$. We seek to generalize the absolute value and the distance to the space $\mathbb{R}^n$. To do so, we will introduce the concepts of a norm and a metric.

Definition 1.1 (The Euclidean norm on $\mathbb{R}^n$)**.** The *Euclidean norm* on $\mathbb{R}^n$ is the function $\|.\|_2 : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \left(\sum_{k=1}^{n} x_k^2 \right)^{\frac{1}{2}}. \tag{1.2}$$

It measures the distance of the point $\mathbf{x}$ to the origin $\mathbf{0} = (0, \ldots, 0)$.

Properties of the Euclidean norm: Observe that in one dimension, the Euclidean norm of a real number is the same as the absolute value of that number. In general, the Euclidean norm satisfies the following properties:

1. **Non-negativity**: $\|\mathbf{x}\|_2 \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$, with equality if and only if $\mathbf{x} = \mathbf{0}$.
2. **Homogeneity**: $\|\lambda \cdot \mathbf{x}\|_2 = |\lambda| \cdot \|\mathbf{x}\|_2$ for all $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.
3. **Triangle inequality**: $\|\mathbf{x} + \mathbf{y}\|_2 \leqslant \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

One of the most important properties of the scalar product is the *Cauchy-Schwarz inequality*, which says that

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leqslant \|\mathbf{x}\|_2 \, \|\mathbf{y}\|_2 \tag{Cauchy-Schwarz}$$

The Euclidean norm $\|\mathbf{x}\|_2$ also corresponds to the length of a vector $\mathbf{x}$. The scalar product $\langle \mathbf{x}, \mathbf{y} \rangle$ measures the angle between the two vectors $\mathbf{x}$ and $\mathbf{y}$: if we designate θ as the angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \cos\theta. \tag{Angle Formula}$$

In particular if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors, i.e., $\theta = \pm\pi/2$, then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. As a consequence, we obtain the famous *Pythagorean theorem*, which says that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal then

$$\|\mathbf{x} + \mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2. \tag{Pythagoras}$$

With the help of the Euclidean norm we can define a metric on $\mathbb{R}^n$ called the Euclidean distance.

Definition 1.2 (The Euclidean distance on $\mathbb{R}^n$)**.** The *Euclidean distance* on $\mathbb{R}^n$ is the function $d(.,.) \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ given by

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_2 = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}. \tag{1.3}$$

Properties of the Euclidean distance: The Euclidean distance captures the natural distance between two points in $\mathbb{R}^n$. It satisfies the following three properties:

1. **Non-negativity**: $d(\mathbf{x}, \mathbf{y}) \geqslant 0$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, with equality only when $\mathbf{x} = \mathbf{y}$.
2. **Symmetry**: $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
3. **Triangle inequality**: $d(\mathbf{x}, \mathbf{y}) \leqslant d(\mathbf{x}, \mathbf{z}) + d(\mathbf{y}, \mathbf{z})$.

1.3 The topology on $\mathbb{R}^n$

The Euclidean distance $d(\mathbf{x}, \mathbf{y})$ induces a topology on $\mathbb{R}^n$ which underpins all analytical considerations on $\mathbb{R}^n$. In particular, notions such as continuity, convergence, differentiablility and integrability are all defined in terms of this topology. The building blocks of the topology on $\mathbb{R}^n$ are the so-called open balls.

Definition 1.3 (Open Ball)**.** Let $\mathbf{a} \in \mathbb{R}^n$ and $r > 0$. The set

$$B(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{a}) < r\}$$

is called the *open ball* of radius r centered at $\mathbf{a}$.

Open balls are the mathematical conceptualization of "nearness" and an important use of open balls is to topologically distinguish distinct points: if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{y}$ then we can find a sufficiently small open ball centered at $\mathbf{x}$ and another sufficiently small open ball centered at $\mathbf{y}$ such that these two balls don't touch.

Open balls are instances of open sets. An open set is a set with the property that if $\mathbf{x}$ is a point in the set then all points that are sufficiently near to $\mathbf{x}$ also belong to the set. The mathematically precise definition is as follows:

Definition 1.4 (Open set)**.** A subset $U \subseteq \mathbb{R}^n$ is *open* if for any point $\mathbf{x} \in U$ there exists $\varepsilon > 0$ such that the open ball $B(\mathbf{x}, \varepsilon)$ is contained in U.

The empty set $\emptyset$ and the space $\mathbb{R}^n$ are open. Also, as was already mentioned, any open ball $B(\mathbf{a}, r)$ is an open set.

Example 1.1 (Open Sets in $\mathbb{R}^n$)**.**

1. If $a < b$ are real numbers then the interval

   $$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

   is an open set. Indeed, if $x \in (a, b)$, simply take $r = \min\{x - a, b - x\}$. Both these numbers are strictly positive, since $a < x < b$, and so is their minimum. Then the "1-dimensional ball" $B(x, r) = \{y \in \mathbb{R} : |x - y| < r\}$ is a subset of (a, b). This proves that (a, b) is an open set.
2. The infinite intervals (a, ∞) and $(-\infty, b)$ are also open but the intervals

   $$(a, b] = \{x \in \mathbb{R} : a < x \leqslant b\} \qquad \text{and} \qquad [a, b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$$

   are not open sets.
3. The rectangle

   $$(a, b) \times (c, d) = \{(x, y) \in \mathbb{R}^2 : a < x < b,\ c < y < d\}$$

   is an open set.

The antithetical notion to an open set is that of a closed set.

Definition 1.5 (Closed set)**.** A subset $C \subseteq \mathbb{R}^n$ is *closed* if its complement $\mathbb{R}^n \backslash C$ is open.

The empty set $\emptyset$ and the space $\mathbb{R}^n$ are the only sets that are both closed and open at the same time. Intuitively, one should think of a closed set as a set that has no "punctures" or "missing endpoints", i.e., it includes all limiting values of points. For instance, the punctured plane $\mathbb{R}^2\backslash\{(0,0)\}$ is not a closed set.

An example of a closed set is the closed ball.

Definition 1.6 (Closed Ball)**.** Let $\mathbf{a} \in \mathbb{R}^n$ and $r > 0$. The set

$$\overline{B(\mathbf{a}, r)} = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{a}) \leqslant r\}$$

is called the closed ball of radius r centered at $\mathbf{a}$. It is a closed set.

Example 1.2 (Closed Sets in $\mathbb{R}^n$)**.**

1. The closed interval

$$[a, b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$$

   is a closed set, because its complement $\mathbb{R}\backslash[a, b] = (-\infty, a) \cup (b, \infty)$ is an open set.
2. Infinite intervals with closed boundary $[a, \infty)$ and $(-\infty, b]$ are closed sets.
3. Halfopen intervals such as $[a, b)$ or $(a, b]$ are neither closed nor open sets.
4. Any set consisting of only finitely many points is a closed set.

The following two propositions describe how open and closed sets behave under basic set manipulations such as unions, intersections, and set differences.

Proposition 1.1.

- *If $U \subseteq \mathbb{R}^n$ is open and $C \subseteq \mathbb{R}^n$ is closed then $U\backslash C$ is open.*
- *If $C \subseteq \mathbb{R}^n$ is closed and $U \subseteq \mathbb{R}^n$ is open then $C\backslash U$ is closed.*

Proposition 1.2.

- *If $U_1, \ldots, U_k \subseteq \mathbb{R}^n$ are open then $U_1 \cup \ldots \cup U_k$ and $U_1 \cap \ldots \cap U_k$ are open.*
- *If $C_1, \ldots, C_k \subseteq \mathbb{R}^n$ are closed then $C_1 \cup \ldots \cup C_k$ and $C_1 \cap \ldots \cap C_k$ are closed.*

To better grasp the difference between open sets and closed sets, we introduce the concept of interior points, exterior points, and boundary points.

Definition 1.7 (Interior, Exterior, Boundary Points)**.** Let S be a subset of $\mathbb{R}^n$ and $\mathbf{x}$ a point in $\mathbb{R}^n$.

(i) We call $\mathbf{x}$ an *interior point* of S if there exists $r > 0$ such that the ball $B(\mathbf{x}, r)$ is contained in S.

(ii) We call $\mathbf{x}$ an *exterior point* of S if there exists $r > 0$ such that the ball $B(\mathbf{x}, r)$ has empty intersection with S.

(iii) We call $\mathbf{x}$ a *boundary point* of S if it is neither an interior point nor an exterior point for S. Equivalently, $\mathbf{x}$ is a boundary point of S if for every $r > 0$ the ball $B(\mathbf{x}, r)$ has non-empty intersection with S without being entirely contained in S.

Note that every point is either interior, exterior or on the boundary in relationship to a set S.

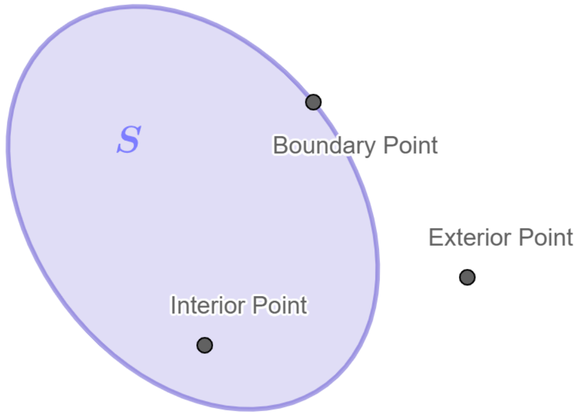


Figure 1.1: Illustration of the difference between interior, exterior and boundary points of a set S.

Definition 1.8 (Interior)**.** The set of all interior points of a set S is called the interior of S and it is denoted by $\mathring{S}$.

Definition 1.9 (Boundary)**.** The set of all boundary points of a set S is called the boundary of S and we use ∂S to denote it.

Definition 1.10 (Closure)**.** The closure of S, denoted by $\overline{S}$, is the set of points $\mathbf{x} \in \mathbb{R}^n$ with the property that for all $r > 0$ one has

$$B(\mathbf{x}, r) \cap S \neq \emptyset.$$

Equivalently, the closure of S is the union of all its interior points and all its boundary points.

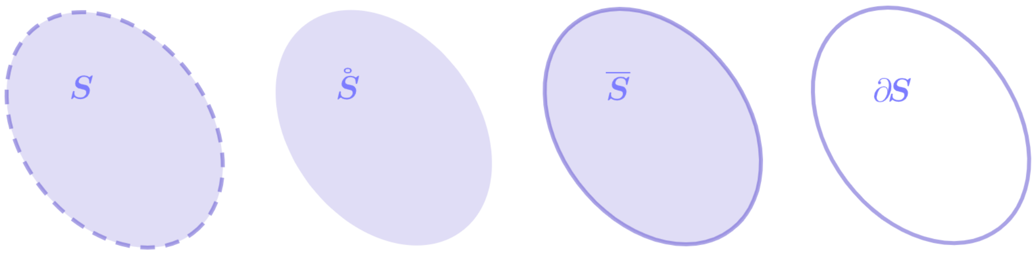


Figure 1.2: The interior, closure and boundary sets of a set S.

Proposition 1.3. *Let $S \subseteq \mathbb{R}^n$. The interior $\mathring{S}$ is the largest open set contained inside of S. The closure $\overline{S}$ is the smallest closed set that has S as a subset.*

Corollary 1.1. *A set is open if and only if it is equal to its interior. On the other hand, a set is closed if and only if it is equal to its closure, which is the same as saying that it contains all its boundary points.*

Properties of closure, interior, and boundary: Suppose $S \subseteq \mathbb{R}^n$.

1. **Closure-interior-boundary relationship**: Clearly, we have the set inclusions
$$\mathring{S} \subseteq S \subseteq \overline{S}.$$
Moreover, the closure of S is S plus its boundary, its interior is S minus its boundary, and the boundary is the closure minus the interior:
$$\mathring{S} = S \backslash \partial S \qquad \overline{S} = S \cup \partial S, \qquad \text{and} \qquad \partial S = \overline{S} \backslash \mathring{S}.$$
2. **Closure of the interior**: The closure of the interior of S is always a subset of the closure of S,
$$\overline{\mathring{S}} \subseteq \overline{S}.$$
This indicates that the closure of the interior of S may capture some but not necessarily all of the boundary ∂S of S.
3. **Interior/closure and complement**: Let $S^c = \mathbb{R}^n \backslash S$ denote the complement of S in $\mathbb{R}^n$. Then
$$\mathring{S^c} = (\overline{S})^c \qquad \text{and} \qquad \overline{S^c} = (\mathring{S})^c.$$
4. **Boundary and complement**: The set S and its complement S^c share the same boundary, i.e.,
$$\partial S = \partial S^c.$$

Example 1.3 (Closure, Interior, Boundary)**.**

1. The sets $(0,1)$, $[0,1]$, $[0,1)$, and $(0,1]$ all have the same closure, interior, and boundary: the closure is $[0,1]$, the interior is $(0,1)$, and the boundary consists of the two points 0 and 1.
2. The sets
$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \qquad \text{and} \qquad \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$$
both have the same closure, interior, and boundary: the closure is the disc of equation $x^2 + y^2 \leqslant 1$, the interior is the disc of equation $x^2 + y^2 < 1$, and the boundary is the circle of equation $x^2 + y^2 = 1$.
3. The set
$$U = \{(x,y) \in \mathbb{R}^2 : |y| < x^2\}$$
describes the region between two parabolas touching at the origin, shown in Fig. 1.3. The set is open, so $U = \mathring{U}$. The closure of U is given by
$$\overline{U} = \{(x,y) \in \mathbb{R}^2 : |y| \leqslant x^2\}.$$
In particular, the closure contains the point $(0,0)$.
4. The unit ball is open in $\mathbb{R}^n$ and is defined by
$$B_1 = B(\mathbf{0}, 1) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < 1\}$$

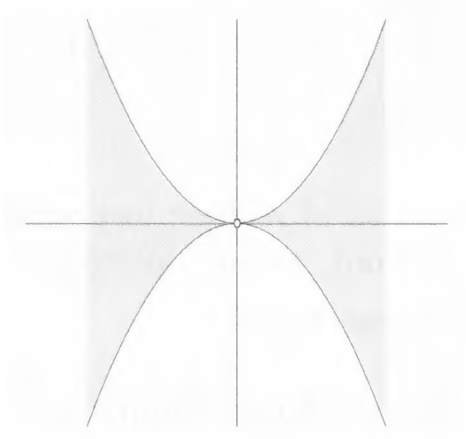


Figure 1.3: The origin belongs to the closure of the shaded region.

Its boundary is the sphere $\partial B_1 = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}$.

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. The set

$$G_f = \{(x, f(x)) \in \mathbb{R}^2 : x \in \mathbb{R}\}$$

is known as the graph of f and represents a curve in $\mathbb{R}^2$. We have $\mathring{G}_f = \emptyset$. Therefore $G_f = \partial G_f$. The closed graph theorem says that graph $\mathring{G}_f$ is a closed set in $\mathbb{R}^2$ if f is a continuous function.

6. Let $B = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 < 1\}$ and $I = [0, 5]$. The set S defined by

$$S = B \times I = \left\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1 \text{ and } 0 \leqslant x_3 \leqslant 5\right\}$$

is a cylinder. The set S is neither closed nor open. The boundary of S is given by

$$\partial S = \underbrace{\partial B \times I}_{E_1} \cup \underbrace{B \times \partial I}_{E_2},$$

where

$$E_1 = \left\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1 \text{ and } 0 \leqslant x_3 \leqslant 5\right\},$$
$$E_2 = \left\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1 \text{ and } x_3 \in \{0, 5\}\right\}.$$

Definition 1.11 (Neighborhood of a point in $\mathbb{R}^n$)**.** Let $\mathbf{x} \in \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$. If $\mathbf{x}$ is an interior point of U then U is called a *neighborhood of* $\mathbf{x}$.

1.4 Sequences in $\mathbb{R}^n$

Limits of sequences and limits of functions are fundamental notions in calculus, as you already have seen in Analysis 1. Let us extend these principles to higher dimensions. We write $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the set of natural numbers.

Definition 1.12 (Sequences in $\mathbb{R}^n$)**.** A *sequence* of elements of $\mathbb{R}^n$ is a function $k \mapsto \mathbf{x}_k$

that associates to every natural number $k \in \mathbb{N}$ an element $\mathbf{x}_k \in \mathbb{R}^n$. We write $(\mathbf{x}_k)_{k\in\mathbb{N}}$ to denote a sequence in $\mathbb{R}^n$.

Although $(\mathbf{x}_k)_{k\in\mathbb{N}}$ is by definition a sequence of n-tuples, we can also think of it as an n-tuple of sequences by considering each coordinate as an individual sequence,

$$(\mathbf{x}_k)_{k\in\mathbb{N}} = \begin{pmatrix} (x_{1,k})_{k\in\mathbb{N}} \\ \vdots \\ (x_{n,k})_{k\in\mathbb{N}} \end{pmatrix}.$$

Definition 1.13 (Convergent sequence)**.** A sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{x} \in \mathbb{R}^n$ if for every $\varepsilon > 0$ there exists $N > 1$ such that when $k \geqslant N$, then $d(\mathbf{x}_k, \mathbf{x}) < \varepsilon$. In this case we call $\mathbf{x}$ the *limit* of $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and write

$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x}.$$

Note that not every sequence has a limit, but if a sequence does then this limit is unique. Sequences that possess a limit are called *convergent*, whereas sequences that don't possess one are called *divergent*.

It follows from Definition 1.13 that a sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{x}$ if and only if the sequence of distances $d(\mathbf{x}_k, \mathbf{x})$ converges to 0, i.e.,

$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x} \quad \Longleftrightarrow \quad \lim_{k\to+\infty} d(\mathbf{x}_k, \mathbf{x}) = 0.$$

Convergence is also observed coordinate wise: A sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{x}$ if and only if each coordinate of $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges to the respective coordinate of $\mathbf{x}$. More precisely, if

$$(\mathbf{x}_k)_{k\in\mathbb{N}} = \begin{pmatrix} (x_{1,k})_{k\in\mathbb{N}} \\ \vdots \\ (x_{n,k})_{k\in\mathbb{N}} \end{pmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

then

$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x} \quad \Longleftrightarrow \quad \lim_{k\to+\infty} x_{i,k} = x_i \ \text{ for all } i = 1, \ldots, n.$$

Example 1.4 (Convergent and divergent sequences in $\mathbb{R}^n$)**.**

1. The sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ given by

$$\mathbf{x}_k = \begin{pmatrix} e^{-k} \\ \frac{k}{k+1} \\ \frac{1}{\sqrt{k^2-k}-k} \end{pmatrix}$$

converges as $k \to +\infty$ to the limit

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix},$$

because $\lim_{k\to+\infty} e^{-k} = 0$, $\lim_{k\to+\infty} \frac{k}{k+1} = 1$, and $\lim_{k\to+\infty} \frac{1}{\sqrt{k^2-k}-k} = -2$.

2. The sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ given by

$$\mathbf{x}_k = \begin{pmatrix} 0 \\ \frac{1-(-1)^k}{2} \end{pmatrix}$$

diverges because it diverges in the second coordinate.

The following properties describe the arithmetic operations of sequences in the n-dimensional Euclidean space and tell us that limits cooperate nicely with the vector space structure of $\mathbb{R}^n$.

Properties of limits of sequences. Let $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\mathbf{y}_k)_{k\in\mathbb{N}}$ be sequences in $\mathbb{R}^n$ and let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

1. **Addition of sequences:** If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\mathbf{y}_k)_{k\in\mathbb{N}}$ both converge then so does $(\mathbf{x}_k+\mathbf{y}_k)_{k\in\mathbb{N}}$ and

$$\lim_{k\to+\infty} \mathbf{x}_k+\mathbf{y}_k = \lim_{k\to+\infty} \mathbf{x}_k + \lim_{k\to+\infty} \mathbf{y}_k.$$

2. **Multiplication of sequences:** If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\lambda_k)_{k\in\mathbb{N}}$ both converge then so does $(\lambda_k\mathbf{x}_k)_{k\in\mathbb{N}}$ and

$$\lim_{k\to+\infty} \lambda_k\mathbf{x}_k = \left(\lim_{k\to+\infty} \lambda_k\right)\cdot\left(\lim_{k\to+\infty} \mathbf{x}_k\right).$$

3. **Inner product of sequences:** If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\mathbf{y}_k)_{k\in\mathbb{N}}$ both converge then so does $(\langle\mathbf{x}_k,\, \mathbf{y}_k\rangle)_{k\in\mathbb{N}}$ and

$$\lim_{k\to+\infty} \langle\mathbf{x}_k,\, \mathbf{y}_k\rangle = \left\langle \lim_{k\to+\infty} \mathbf{x}_k,\, \lim_{k\to+\infty} \mathbf{y}_k \right\rangle.$$

Definition 1.14 (Cauchy sequences)**.** A sequence $(\mathrm{x}_k)_{k\in\mathbb{N}}$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N > 1$ such that $k, l \geqslant N$ implies $d(\mathbf{x}_k, \mathbf{x}_l) < \varepsilon$.

Theorem 1.1. *Every convergent sequence $(\mathrm{x}_k)_{k\in\mathbb{N}}$ is a Cauchy sequence and every Cauchy sequence is convergent.*

Proposition 1.4. *Let $S \subseteq \mathbb{R}^n$ be a non-empty set and suppose $\mathbf{x} \in \partial S$ is a boundary point of S. Then there exists a sequence of elements in $\mathring{S}$, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots \in \mathring{S}$, such that*

$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x}.$$

The following example provides an illustration of the content of Proposition 1.4.

Example 1.5. Consider the open ball of radius 5 centered at the origin in $\mathbb{R}^2$,

$$B(\mathbf{0}, 5) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 < 5\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 25\}.$$

The boundary of $B((0, 0), 5)$ is the circle of radius 5 centered at the origin, i.e.,

$$\partial B(\mathbf{0}, 5) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 = 5\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 25\}.$$

Any point $\mathbf{x} \in \partial B(\mathbf{0}, 5)$ of this circle takes the form

$$\mathbf{x} = \begin{pmatrix} 5\cos\theta \\ 5\sin\theta \end{pmatrix}, \quad \text{for some } \theta \in [0, 2\pi).$$

We can define a sequence

$$\mathbf{x}_k = \begin{pmatrix} \frac{5k}{k+1}\cos\theta \\ \frac{5k}{k+1}\sin\theta \end{pmatrix},$$

and note that $\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x}$. So we see that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a sequence of points inside the open ball $B(\mathbf{0}, 5)$ converging to the point $\mathbf{x}$ on the border .

Proposition 1.5. *Let $S \subseteq \mathbb{R}^n$ be a non-empty subset of $\mathbb{R}^n$ and let $(\mathbf{x}_k)_{k\in\mathbb{N}}$ be a sequence of elements in S. If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges then the limit $\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x}$ must belong to $\overline{S}$, the closure of S.*

Example 1.6. Consider the “halfopen” rectangle

$$S = [0, 1] \times [0, 1).$$

This is not a closed set, because the point $(\frac{2}{3}, 1)$, for example, is in the boundary ∂S but not in S itself. Moreover, the sequence

$$\begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{3}{4} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{4}{5} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{5}{6} \end{pmatrix}, \ldots$$

is a sequence of points in the interior of S that converge to the point $(\frac{2}{3}, 1)$, which is not part of S, but it is part of the closure of S.

Definition 1.15 (Bounded set)**.** A subset $E \subseteq \mathbb{R}^n$ is *bounded* if it is contained in a ball of finite radius centered at the origin:

$$E \subseteq B(\mathbf{0}, R) \quad \text{for some } R < \infty.$$

Note that a closed set need not be bounded. For instance, the interval $[0, \infty)$ is closed, but it is not a bounded.

Definition 1.16 (Compact set)**.** A subset $C \subseteq \mathbb{R}^n$ is *compact* if it is closed and bounded.

Compactness is the basic "finiteness criterion" for subsets of $\mathbb{R}^n$. An important characterization of compact sets in Euclidean spaces is given by the Bolzano-Weierstrass theorem. Before we can state this theorem, we need to recall what is a subsequence.

Definition 1.17 (Subsequence)**.** A *subsequence* of a sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ is any sequence of the form $(\mathbf{x}_{k_i})_{i\in\mathbb{N}}$, where $(k_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence of positive integers.

If a sequence converges then any subsequence of it also converges to the same limit.

Theorem 1.2 (Bolzano-Weierstrass theorem in $\mathbb{R}^n$)**.** *Let $C \subseteq \mathbb{R}^n$ be compact. Any sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ of elements in C possesses a convergent subsequence $(\mathbf{x}_{k_i})_{i\in\mathbb{N}}$ whose* *limit is in C.*

Definition 1.18 (Bounded sequences in $\mathbb{R}^n$)**.** A sequence $(\mathrm{x}_k)_{k\in\mathbb{N}}$ is *bounded* if there exists a constant $C > 0$ such that $\|\mathbf{x}_k\|_2 \leqslant C$ for any $k \in \mathbb{N}$.

Note that every convergent sequence is a bounded sequence, but the opposite is in general not true. For example, the sequence $x_k = (-1)^k$ is bounded and does not converge. The following is an immediate corollary of the Bolzano-Weierstrass theorem.

Corollary 1.2. *Each bounded sequence* $(\mathrm{x}_k)_{k\in\mathbb{N}}$ *in* $\mathbb{R}^n$ *has a convergent subsequence* $(\mathbf{x}_{k_i})_{i\in\mathbb{N}}$.

Chapter 2

Real-valued functions in $\mathbb{R}^n$

Multivariable calculus, also known as *multivariate calculus*, is the extension of calculus in one variable to calculus with functions of several variables. We start by defining real-valued functions in more than one variable.

2.1 Definition

Definition 2.1 (Real-valued function on $E \subseteq \mathbb{R}^n$)**.** Let E be a non-empty subset of $\mathbb{R}^n$. A function $f\colon E \to \mathbb{R}$ that assigns to every element $\mathbf{x} \in E$ a unique real number $y = f(\mathbf{x})$ is called a *real-valued function* on E.

Given a function $f\colon E \to \mathbb{R}$, the *domain* of f is E, denoted $\operatorname{dom}(f)$ or $\operatorname{dom} f$. In theory, the domain should always be a part of the definition of the function rather than a property of it, but in practice it is often the case that the domain is inferred by the description of the function (see Examples 2.1 and 2.3 below).

The *image* (sometimes also called the *range*) of a function f is the set of all the output values that f produces. We denote it by $\operatorname{Im}(f)$ and it is formally defined as

$$\operatorname{Im}(f) = \{f(\mathbf{x}) : \mathbf{x} \in E\} = \{y \in \mathbb{R} : \exists \mathbf{x} \in E \text{ with } f(\mathbf{x}) = y\}.$$

Example 2.1. Let us find and sketch the domain of the function

$$f(x, y) = \frac{\sqrt{x + y + 1}}{(x - 1)}.$$

The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is:

$$\operatorname{dom}(f) = \{(x, y) \in \mathbb{R}^2 : x + y + 1 \geqslant 0, x \neq 1\}.$$

The inequality $x + y + 1 > 0$, or $y > -x - 1$, describes the points that lie on or above the line $y = -x - 1$, while $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain. See Fig. 2.1 for a sketch of this region.

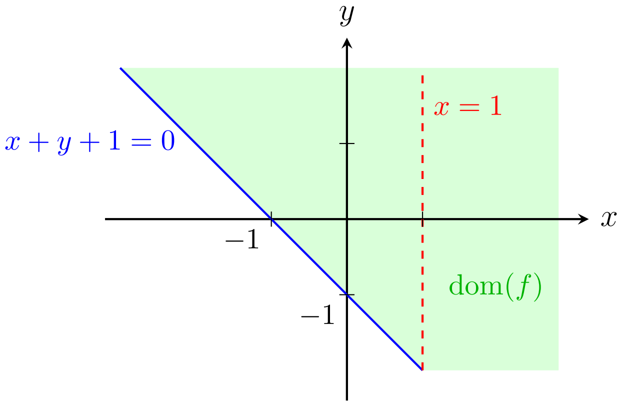


Figure 2.1: The domain of the function $f(x, y) = \frac{\sqrt{x+y+1}}{(x-1)}$
.

The relationship between the domain and the image of a function is described by its *graph*. We use $G(f)$ to denote the graph of a function $f : E \to \mathbb{R}$ and it is given by

$$G(f) = \left\{ \begin{pmatrix} \mathbf{x} \\ f(\mathbf{x}) \end{pmatrix} : \mathbf{x} \in D \right\} \subseteq \mathbb{R}^{n+1}.$$

Note that the graph of f is a subset of $\mathbb{R}^{n+1}$. More precisely, the graph is the hypersurface in $\mathbb{R}^{n+1}$ corresponding to the set of all points $(x_1, \dots, x_n, x_{n+1})^\top \in \mathbb{R}^{n+1}$ that satisfy the equation

$$x_{n+1} = f(x_1, \dots, x_n).$$

Example 2.2. Consider the equation $x + y = z$; as you learned in linear algebra, the solutions to this equation describe a plane in $\mathbb{R}^3$. Now, compare this with the function $f(x, y) = x + y$, a real-valued function in two variables. By definition, the graph of $f(x, y)$ consists of points $(x, y, z) \in \mathbb{R}^3$ where $z = f(x, y)$. For $f(x, y) = x + y$, this gives the equation of the plane $x + y = z$. Thus, the graph of $f(x, y) = x + y$ is exactly the plane in $\mathbb{R}^3$ determined by the equation $x + y = z$.

Example 2.2 connects what you studied in linear algebra, where you worked with linear equations like $x + y = z$, to what you're learning now in multivariable calculus. But there's more! With multivariable functions, you can describe not just planes, but much more complex geometric surfaces, as this next example illustrates.

Example 2.3. Consider the real-valued function $f(x, y) = \sqrt{1 - x^2 - y^2}$, which is a function in 2 variables. The natural domain of this function is $\mathrm{dom}(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$, which is the closed disc of radius 1 centered at the origin. The image of f is $\mathrm{Im}(f) = [0, 1]$ and the graph $G(f) = \{(x, y, z) \in D \times \mathbb{R}, z = f(x, y)\}$ coincides with the set of solutions to the equations

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad z \geqslant 0.$$

In other words, the graph of the function is a *semi-sphere*, see Fig. 2.2 below.

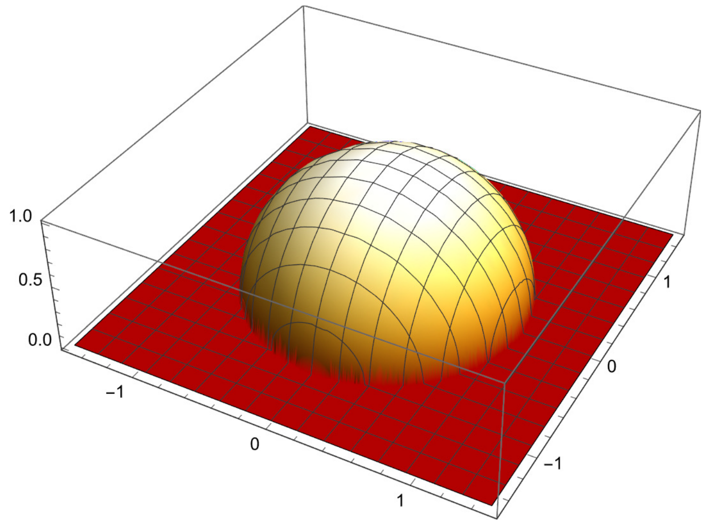


Figure 2.2: Graph of the function $f(x, y) = \sqrt{1 - x^2 - y^2}$.

Example 2.4. In physics, the functions $f\colon \mathbb{R}^n \to \mathbb{R}$ are often called scalar fields. The gravitational potential of a mass or the electric potential of an electric charge are examples of scalar fields:

$$\phi : \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{R}, \quad \phi(\mathbf{x}) = \frac{k}{\|\mathbf{x}\|_2}$$

for a real constant k. In mechanics, we often consider systems where the energy is conserved (Hamiltonian systems). For the movement of a particle of mass m in space, subject to the potential $V(\mathbf{x})$, its energy is a real-valued function of its momentum $\mathbf{p} = m\mathbf{v}$ here $\mathbf{v}$ is the velocity and $\mathbf{x}$ the position in space:

$$E : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad E(\mathbf{p}, \mathbf{x}) = \frac{\|\mathbf{p}\|_2^2}{2m} + V(\mathbf{x}).$$

The movement follows the lines at which the energy E is constant. These lines are called “contour lines” and they are special cases of so-called *level sets*, which we define and discuss next.

2.2 Level Sets

Definition 2.2 (Level set)**.** Let $f\colon E \to \mathbb{R}, E \subseteq \mathbb{R}^n (E \neq \emptyset)$. Given a real number $c \in \mathrm{Im}(f)$, we call the set

$$L_c(f) = \{\mathbf{x} \in D : f(\mathbf{x}) = c\} = f^{-1}(\{c\})$$

the *level set* of f at height c. If $c \notin Im(f)$, then $L_c(f) = \emptyset$.

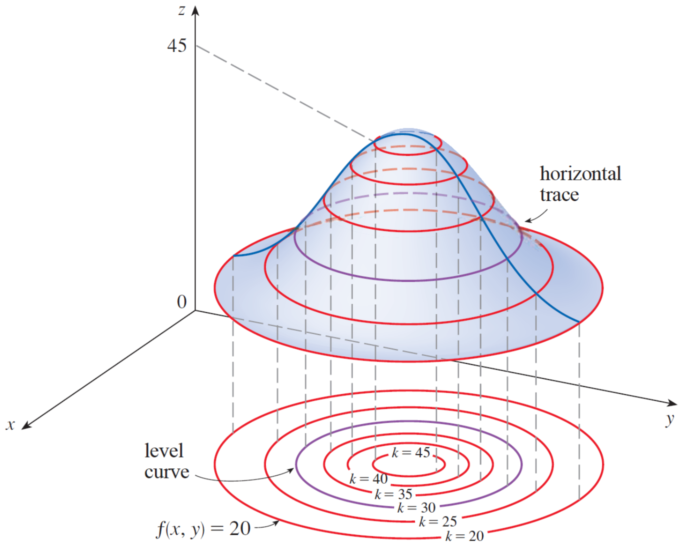


Figure 2.3: The figure displays the graph of a function in 2 variables together with an illustration of its level curves in the xy-plane. One can also think of level curves as the projection of the horizontal traces onto the xy-plane, where a *horizontal trace* is a line formed by intersecting the graph of the function with a plane parallel to the xy-plane.

Level sets of functions in 2 variables $f\colon \mathbb{R}^2 \to \mathbb{R}$ are sometimes also called *level curves* (or *contour lines*). It represents all the points where f has *"height" c*. A collection of contour lines is called a *contour map*. Contour maps are very helpful for visualizing functions, and they are most descriptive if the level curves are drawn for equally spaced heights, see Fig. 2.4.

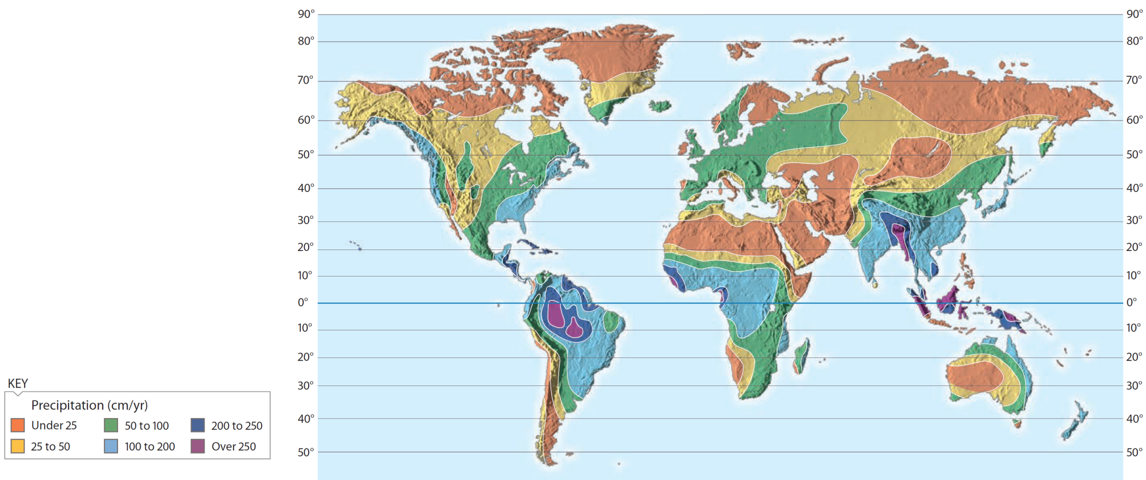


Figure 2.4: Contour map of participation as a function in two variables, the longitude and latitude coordinates on earth.

In summary, we now have learned of two ways of graphically representing a real-valued functions in two variables. The first way is by its graph, which is a hypersurface in $\mathbb{R}^3$, and the second is by a contour map, the projection of its contour lines onto the plane $\mathbb{R}^2$. In Fig. 2.5 below you can see these two methods juxtaposed.

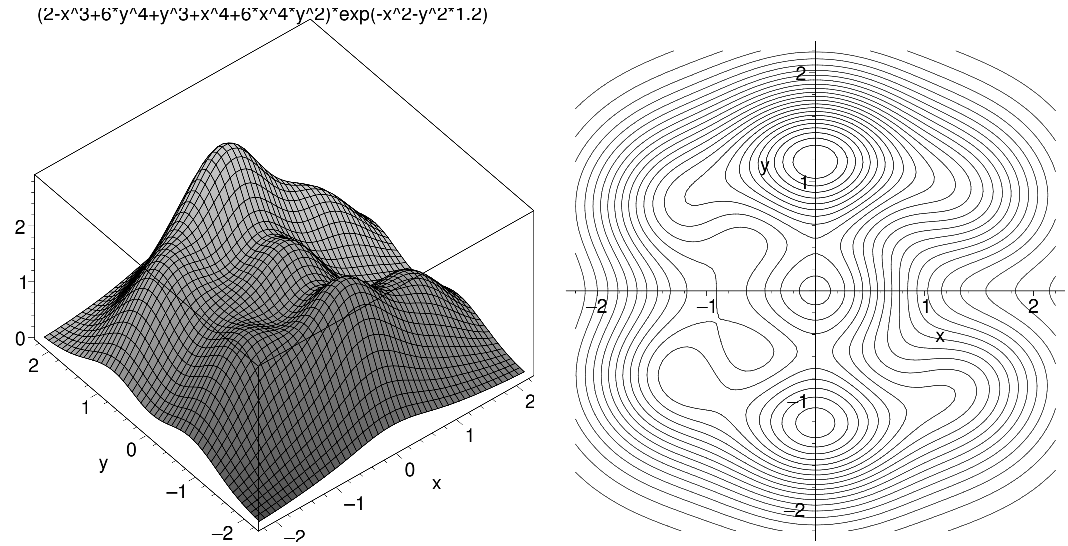


Figure 2.5: Depiction of graph (left) and contour diagram (right) of the same function in 2 variables.

Example 2.5. Let $f(x, y) = \frac{xy-1}{\sqrt{y-x^2}}$, whose domain is $\text{dom}(f) = \{(x, y) \in \mathbb{R}^2 : y > x^2\}$. Notice that $\text{dom}(f)$ is open and unbounded.

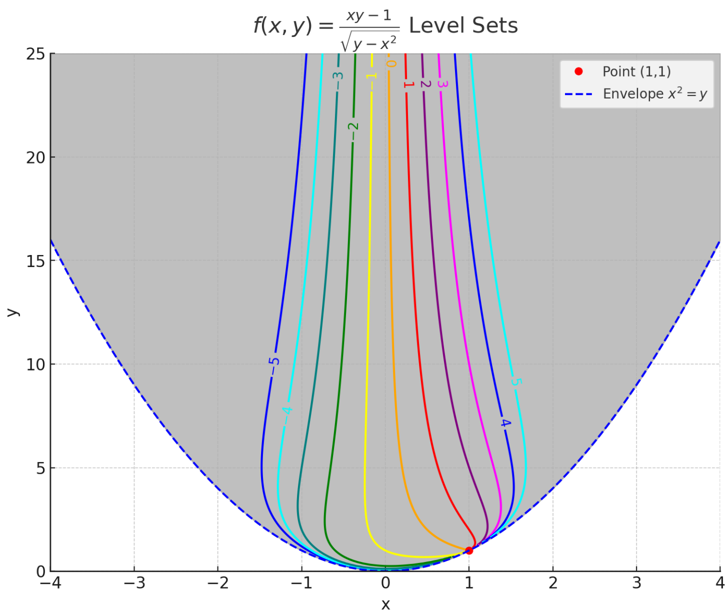


Figure 2.6: The figure displays a series of level curves for the function $f(x, y) = \frac{xy-1}{\sqrt{y-x^2}}$ passing through the point $(1, 1)$. As we will explore subsequently, this indicates that the limit of $f(x, y)$ as (x, y) approaches (1,1) is not well-defined.

2.3 Limits of functions

Definition 2.3. Let $f\colon E \to \mathbb{R}$ with $E \subseteq \mathbb{R}^n$. We say that f is *defined in a neighborhood of* $\mathbf{x}_0 \in \mathbb{R}^n$ if $\mathbf{x}_0$ is an interior point of $E \cup \{\mathbf{x}_0\}$. That is, there exists $\delta > 0$ such that $B(\mathbf{x}_0, \delta) \subseteq E \cup \{\mathbf{x}_0\}$.

In the above definition, it is possible that $\mathbf{x}_0 \notin E$. In other words, it is possible for a function to be defined in a neighborhood of $\mathbf{x}_0 \in \mathbb{R}^n$ without being defined at $\mathbf{x}_0$ itself.

Example 2.6. Consider the function $f(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|}$ whose domain equals $\operatorname{dom}(f) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \neq 0\} = \mathbb{R}^n \backslash \{\mathbf{0}\}$. Although this function is not defined at $\mathbf{0}$, it is defined in a neighborhood of $\mathbf{0}$.

We are concerned with points where the function is defined in a neighborhood around the point, because this is necessary to properly define the limit of a function at that point. If the function is not defined in the neighborhood of a point, then it is not always possible to talk about the limit of the function at that point without running into mathematical contradictions.

Definition 2.4 (Limit of a function)**.** Let E be a subset of $\mathbb{R}^n$, $f\colon E \to \mathbb{R}$ a function with domain E and assume f is defined in a neighborhood of the point $\mathbf{x}_0 \in \mathbb{R}^n$. We say that f has a *limit* $l \in \mathbb{R}$ at $\mathbf{x}_0$ and write

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = l,$$

if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x} \in E$,

$$0 < d(\mathbf{x}, \mathbf{x}_0) \leqslant \delta \implies |f(\mathbf{x}) - l| \leqslant \varepsilon$$

Note that the limit of a function at a point does not always exist. But if it does exists then it is unique, which means that a function has at most one limit at a given point.

Example 2.7. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Let's calculate its limit as (x, y) approaches $(0, 0)$. We will learn several different methods of finding the limit of a function at a point (see, for example, the Squeeze Theorem below), but the most standard method consists of simply verifying Definition 2.4. Given the relation $0 \leqslant \sqrt{x^2+y^2}$, we have

$$\begin{aligned}|f(x,y)| &= \frac{|x+y|\,|x^2-xy+y^2|}{x^2+y^2} \leqslant (|x|+|y|)\frac{x^2+|x||y|+y^2}{x^2+y^2} \\ &\leqslant (|x|+|y|)\frac{x^2+|x||y|+y^2+\frac{1}{2}(|x|-|y|)^2}{x^2+y^2} \\ &= (|x|+|y|)\frac{\frac{3}{2}x^2+\frac{3}{2}y^2}{x^2+y^2} \\ &\leqslant 2\sqrt{x^2+y^2}\frac{\frac{3}{2}x^2+\frac{3}{2}y^2}{x^2+y^2} = 3\sqrt{x^2+y^2} = 3\|(x,y)\|_2.\end{aligned}$$

This shows that as long as $\delta < \frac{\varepsilon}{3}$ we have $d((x,y),(0,0)) < \delta \implies |f(x,y)| \leqslant \varepsilon$. According to Definition 2.4, this means exactly that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Proposition 2.1 (Characterization of limits by sequences)**.** *Let $E \subseteq \mathbb{R}^n, \mathbf{x}_0 \in \mathbb{R}^n$ and assume $f\colon E \to \mathbb{R}$ defined on a neighbourhood of $\mathbf{x_0}$, and $l \in \mathbb{R}^n$. The following statements are equivalent:*

1. $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = l$.
2. $\lim_{k\to\infty} f(\mathbf{x}_k) = l$ *for every sequence* $(\mathbf{x}_k)_{k\in\mathbb{N}}$ *in* $E\backslash\{\mathbf{x}_0\}$ *with* $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}_0$.

Properties of limits of functions. Let f and g be two functions defined in a neighborhood of $\mathbf{x}_0$ and assume the limits $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})$ exist.

1. **Linearity**: For constants $\alpha, \beta \in \mathbb{R}$, we have

$$\lim_{\mathbf{x}\to\mathbf{x}_0}(\alpha f(\mathbf{x}) + \beta g(\mathbf{x})) = \alpha\Big(\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})\Big) + \beta\Big(\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})\Big)$$

2. **Products**:

$$\lim_{\mathbf{x}\to\mathbf{x}_0}(f(\mathbf{x}) \cdot g(\mathbf{x})) = \Big(\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})\Big) \cdot \Big(\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})\Big).$$

3. **Quotients**: If $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) \neq 0$, then

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\left(\frac{f(\mathbf{x})}{g(\mathbf{x})}\right) = \frac{\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})}.$$

4. **Compositions**: Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ be given. If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exists, and $g_i : \mathbb{R} \to \mathbb{R}$ are functions such that $\lim_{x\to b_i} g_i(x) = a_i$ for each i, then

$$\lim_{\mathbf{x}\to\mathbf{b}} f(g_1(x_1), g_2(x_2), \ldots, g_n(x_n)) = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}).$$

Example 2.8. Let us calculate

$$\lim_{(x,y)\to(-3,4)} \frac{1+xy}{1-xy}.$$

Since $\lim_{(x,y)\to(-3,4)} x = -3$ and $\lim_{(x,y)\to(-3,4)} y = 4$, it follows from properties 1 and 2 of limits of functions that

$$\lim_{(x,y)\to(-3,4)} 1+xy \;=\; 1+\Big(\lim_{(x,y)\to(-3,4)} x\Big)\Big(\lim_{(x,y)\to(-3,4)} y\Big) \;=\; 1+(-3)\cdot 4 = -11.$$

Similarly, we obtain $\lim_{(x,y)\to(-3,4)} 1-xy = 13$. Since the limit of the numerator and denominator exist and the denominator does not converge to 0, it follows from property 3 of limits of functions that

$$\lim_{(x,y)\to(-3,4)} \frac{1+xy}{1-xy} \;=\; \frac{\lim_{(x,y)\to(-3,4)} 1+xy}{\lim_{(x,y)\to(-3,4)} 1-xy} \;=\; -\frac{11}{13}.$$

2.4 Techniques for finding limits of functions

Example 2.9 (The problem with limits in several variables)**.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function in two variables; we would like to determine the limit

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

A (naïve) idea is to compute the two iterated limits:

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) \qquad \text{or} \qquad \lim_{y\to 0}\lim_{x\to 0} f(x,y).$$

If these two limits exist and coincide, one might then be led to believe that the limit of the function at $(0,0)$ is equal to 0. However, <u>this is note true</u>! For example, consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y)\neq(0,0),\\ 0, & \text{if } (x,y)=(0,0). \end{cases}$$

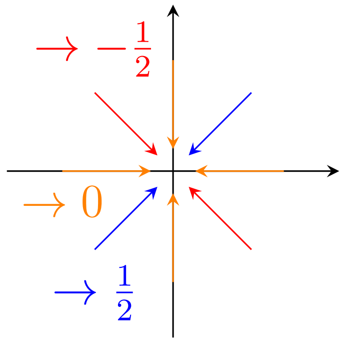
For this particular function, we find that the iterated limits are:

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0}\frac{xy}{x^2+y^2} = \lim_{x\to 0}\frac{0}{x^2+0} = 0,$$
$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0}\frac{xy}{x^2+y^2} = \lim_{y\to 0}\frac{0}{0+y^2} = 0.$$

However, instead having the two variables approach 0 one after the other, we can have them approach zero simultaneously, for example along the diagonal $x=y$. In this case, setting both x and y equal to t and letting t go to zero, we obtain

$$\lim_{t\to 0} f(t,t) = \lim_{t\to 0}\frac{t\cdot t}{t^2+t^2} = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2},$$

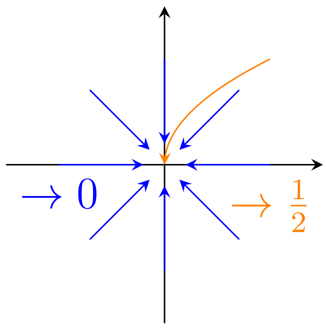
which yields a different result. Since we can approach $(0,0)$ in two different ways and obtain different results, it means that the limit does not exist.



A next idea would be to test all possible directions,

$$\lim_{t \to 0} f(\alpha t, \beta t),$$

with $\alpha, \beta \in \mathbb{R}$ not both zero (thus covering all lines of equation $\beta x - \alpha y = 0$, which are all lines passing through 0). If all the limits along all the lines passing through 0 exist and coincide, can we conclude that the limit exists? The answer is still no! This is because we might obtain a different result when following a trajectory that is not a straight line.



For example, if $f \colon \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

then for any $\alpha, \beta \in \mathbb{R}$, we have

$$\lim_{t \to 0} f(\alpha t, \beta t) = \lim_{t \to 0} \frac{\alpha \beta^2 t^3}{\alpha^2 t^2 + \beta^4 t^4}.$$

If $\alpha = 0$, then $\beta \neq 0$ and we obtain 0. Otherwise,

$$\lim_{t \to 0} f(\alpha t, \beta t) = \lim_{t \to 0} \frac{\alpha \beta^2 t}{\alpha^2 + \beta^4 t^2} = \frac{0}{\alpha + 0} = 0.$$

We obtain 0 in all directions. However,

$$\lim_{t \to 0} f(t^2, t) = \lim_{t \to 0} \frac{t^4}{t^4 + t^4} = \frac{1}{2}.$$

Again, this means that the limit does not exist.

2.4.1 The squeeze theorem

Theorem 2.1 (Squeeze theorem - théorème des gendarmes)**.** *Let $E \subseteq \mathbb{R}^n$, and functions $f, g, h : E \to \mathbb{R}$ be defined on a neighborhood of $\mathbf{x}_0 \in \mathbb{R}^n$. If*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{x}_0} h(\mathbf{x}) = l$$

and there exists $\varepsilon > 0$ such that for all $\mathbf{x} \in E$,

$$0 < d(\mathbf{x}, \mathbf{x}_0) < \varepsilon \implies g(\mathbf{x}) \leqslant f(\mathbf{x}) \leqslant h(\mathbf{x})$$

then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = l.$$

Example 2.10. Consider $f\colon \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ defined by

$$f(x,y) = \frac{x^4y^3}{x^4 + y^{12}}.$$

Let's discuss the limit

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

We can estimate

$$0 \leqslant f(x,y) = \frac{x^4y^3}{x^4+y^{12}} \leqslant \frac{x^4y^3}{x^4} = y^3.$$

So if we define

$$g(x,y) = 0 \qquad \text{and} \qquad h(x,y) = y^3$$

then $g(x,y) \leqslant f(x,y) \leqslant h(x,y)$. Since $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} h(x,y) = 0$, it follows from the Squeeze Theorem that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

2.4.2 Using Polar coordinates

Polar coordinates are useful when given a function in two variables involving terms like $x^2 + y^2$, representing the distance from the origin, or when the function behaves similarly along all directions (i.e., has radial symmetry). This simplifies the analysis by converting the problem into one of radial distance and angular symmetry, making it easier to evaluate limits as the distance from the origin approaches zero.

The following version of the squeeze theorem involving polar coordinates allows us to bound a function in terms of its distance from the origin, making it easier to evaluate limits as the distance approaches zero.

Theorem 2.2 (Squeeze theorem in polar coordinates)**.** *Let $E \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$. Assume $f\colon E \to \mathbb{R}$ is a function that is defined in the neighborhood of (x_0, y_0) and let*

$l \in \mathbb{R}$. Then,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = l$$

if and only if there exists $\varepsilon > 0$ and a function $\psi\colon (0,\varepsilon) \to [0,\infty)$ such that

(i) $\lim_{r\to 0^+} \psi(r) = 0$, and

(ii) for all $\theta \in [0, 2\pi)$ we have $|f(x_0 + r\cos\theta, y_0 + r\sin\theta) - l| \leqslant \psi(r)$

Example 2.11. Consider $f\colon \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ defined by

$$f(x,y) = \frac{x^2 y}{x^2 + y^{\frac{5}{2}}}.$$

Let's discuss the limit

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

We switch to polar coordinates and get

$$\begin{aligned} f(r\cos\theta, r\sin\theta) &= \frac{r^3\cos^2\theta\sin\theta}{r^2\cos^2\theta + r^{\frac{5}{2}}\sin^{\frac{5}{2}}\theta} \\ &= \frac{r\cos^2\theta\sin\theta}{\cos^2\theta + r^{\frac{1}{2}}\sin^{\frac{5}{2}}\theta}. \end{aligned}$$

Thus,

$$|f(r\cos\theta, r\sin\theta)| = \frac{r\cos^2\theta|\sin\theta|}{\cos^2\theta + r^{\frac{1}{2}}\sin^{\frac{5}{2}}\theta} \leqslant \frac{r\cos^2\theta|\sin\theta|}{\cos^2\theta} = r|\sin\theta| \leqslant r.$$

Taking $l = 0$ and $\psi(r) = r$, we see that the hypothesis of the squeeze theorem in polar coordinates is satisfied, and conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

2.4.3 Using Taylor's theorem

Taylor's theorem (which you have learned in Analysis I) can be useful to find limits because it approximates a function near a point by a polynomial, simplifying the analysis before applying the squeeze theorem. For convenience, let us quickly recall the statement of Taylor's theorem.

Theorem 2.3 (Taylor's theorem – single variable case)**.** *Let $k \in \mathbb{N}$. Suppose $I \subseteq \mathbb{R}$ is an open interval and $f\colon I \to \mathbb{R}$ is a function of class $C^k(I)$. Then for any $a \in I$ we* *have*

$$f(x) = \underbrace{\overbrace{\sum_{j=1}^{k} \frac{f^{(j)}(a)}{j!}(x-a)^k}^{k^{\text{th}}\text{-order approximation}} + \overbrace{r_k(x)}^{\text{remainder}}}_{k^{\text{th}}\text{-order expansion}}$$

where $r_k(x)$ is an "error" term satisfying $\lim_{x\to a} \frac{r_k(x)}{|x-a|^k} = 0$.

Example 2.12. Calculate the following limits if they exist:

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2+\ln\left(1+y^2\right)}{\sqrt{x^2+y^2}}$

(b) $\lim_{(x,y)\to(0,0)} \frac{1-e^{(x^3)}}{x^2+y^2}$

(a) The first-order expansion of $\ln(1+x)$ around $a=0$ is

$$\ln(1+x) = x + r_1(x)$$

where $\lim_{x\to 0} \frac{r_1(x)}{x} = 0$. We obtain

$$\lim_{(x,y)\to(0,0)} \frac{x^2+\ln(1+y^2)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2+r_1(y^2)}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2}} + \lim_{(x,y)\to(0,0)} \frac{r_1(y^2)}{\sqrt{x^2+y^2}} = 0+0 = 0.$$

The second limit is zero because, for $(x,y) \neq (0,0)$,

$$-\frac{|r_1(y^2)|}{|y|} \leqslant \frac{r_1(y^2)}{\sqrt{x^2+y^2}} \leqslant \frac{|r_1(y^2)|}{|y|}$$

with

$$\lim_{(x,y)\to(0,0)} \frac{|r_1(y^2)|}{|y|} = \lim_{(x,y)\to(0,0)} |y| \cdot \lim_{(x,y)\to(0,0)} \frac{|r_1(y^2)|}{|y^2|} = 0\cdot 0 = 0.$$

By the squeeze theorem, it follows that

$$\lim_{(x,y)\to(0,0)} \frac{r_1(y^2)}{\sqrt{x^2+y^2}} = 0.$$

(b) The first-order expansion of e^x around $a=0$ is

$$e^x = 1 + x + r_1(x)$$

where $\lim_{x\to 0} \frac{r_1(x)}{x} = 0$. We obtain

$$\lim_{(x,y)\to(0,0)} \frac{1-e^{x^3}}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{1-1-x^3-r_1(x^3)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{-x^3-r_1(x^3)}{x^2+y^2}.$$

Now, for $(x, y) \neq (0, 0)$,

$$-\frac{|x^3| + |r_1(x^3)|}{|x^2|} \leqslant \frac{-x^3 - r_1(x^3)}{x^2 + y^2} \leqslant \frac{|x^3| + |r_1(x^3)|}{|x^2|}$$

with

$$\lim_{(x,y)\to(0,0)} \frac{|x^3| + |r_1(x^3)|}{|x^2|} = \left(\lim_{(x,y)\to(0,0)} \frac{|x^3|}{|x^2|}\right) + \left(\lim_{(x,y)\to(0,0)} \frac{|r_1(x^3)|}{|x^2|}\right) = 0 + 0 = 0.$$

The squeeze theorem therefore ensures that

$$\lim_{(x,y)\to(0,0)} \frac{-x^3 - r_1(x^3)}{x^2 + y^2} = 0.$$

2.4.4 Using change of variables

The following proposition enables us to convert limits in two variables into limits in a single variable.

Proposition 2.2 (Composition with Functions of a Single Variable)**.** *Let $E \subseteq \mathbb{R}^2$ and let $g\colon E \to \mathbb{R}$ be defined in a neighborhood of $(x_0, y_0) \in \mathbb{R}^2$. Let $I \subseteq \mathbb{R}$ be such that $I \subseteq g(E)$ and let $\varphi\colon I \to \mathbb{R}$ be defined in a neighborhood of $l \in \mathbb{R}$. Finally, let $f\colon E \to \mathbb{R}$ be defined by $f(x, y) = \varphi(g(x, y))$. If*

$$\lim_{(x,y)\to(x_0,y_0)} g(x, y) = l \qquad \textit{and} \qquad \lim_{t\to l} \varphi(t) \textit{ exists,}$$

then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \lim_{t\to l} \varphi(t).$$

Example 2.13. Let $f\colon \mathbb{R}^2\backslash\{(0, 0)\} \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{\tan(3x^2 + y^2)}{3x^2 + y^2}.$$

We analyze the limit

$$\lim_{(x,y)\to(0,0)} f(x, y).$$

If we define $g(x, y) = 3x^2 + y^2$, then by properties of limits we have

$$\lim_{(x,y)\to(0,0)} g(x, y) = 3\Big(\lim_{(x,y)\to(0,0)} x\Big)^2 + \Big(\lim_{(x,y)\to(0,0)} y\Big)^2 = 3 \cdot 0^2 + 0^2 = 0.$$

Define $\varphi\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$ by

$$\varphi(t) = \frac{\tan(t)}{t}.$$

Then we have $f(x, y) = \varphi(g(x, y))$. Hence, in light of Proposition 2.2, we have

$$\lim_{(x,y)\to(0,0)} \frac{\tan(3x^2 + y^2)}{3x^2 + y^2} = \lim_{t\to 0} \frac{\tan(t)}{t}.$$

Now,

$$\lim_{t\to 0} \frac{\tan(t)}{t} \overset{\text{L'Hôpital's Rule}}{=} \lim_{t\to 0} \frac{\frac{1}{\cos^2(t)}}{1} = 1.$$

Thus,

$$\lim_{(x,y)\to(0,0)} f(x, y) = 1.$$

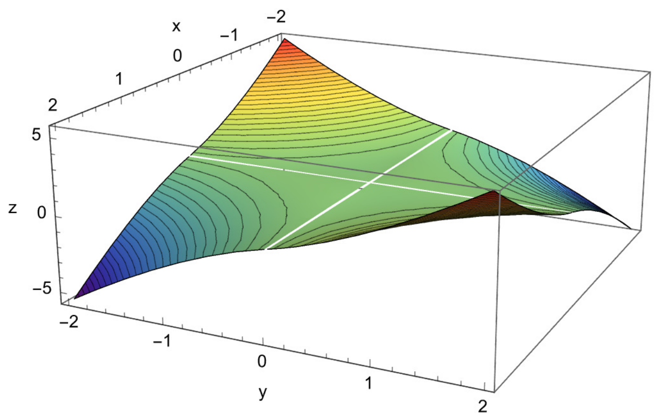


Figure 2.7: Graph of the function $f(x, y) = xy \ln(|x| + |y|)$.

Example 2.14. Let us demonstrate that the limit of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} xy \ln(|x| + |y|) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is zero as (x, y) approaches $(0, 0)$ (see Fig. 2.7). Note that for every point (x, y) with $0 < \sqrt{x^2 + y^2} < 1$ we have $|xy| \leqslant |x| + |y|$. This implies that for any such (x, y) we have the estimate

$$0 \leqslant |f(x, y)| = |xy \ln(|x| + |y|)| \leqslant (|x| + |y|)|\ln(|x| + |y|)|.$$

So if we define

$$g(x, y) = -(|x| + |y|)|\ln(|x| + |y|)| \qquad \text{and} \qquad h(x, y) = (|x| + |y|)|\ln(|x| + |y|)|$$

then we see that

$$0 < \sqrt{x^2 + y^2} < 1 \implies g(x, y) \leqslant f(x, y) \leqslant h(x, y).$$

Substituting t for $|x|+|y|$, it follows from Proposition 2.2 that:

$$\lim_{(x,y)\to(0,0)} \pm(|x|+|y|)|\ln(|x|+|y|)| = \lim_{t\to 0+} t\ln t = 0,$$

where we used the fact $\lim_{t\to 0+} t\ln t = 0$, which can be verified using L'Hôpital's Rule. In other words $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} h(x,y) = 0$. Invoking the Squeeze Theorem, we conclude that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

2.4.5 Testing along polynomial paths

Testing paths of the form (t^α, t^β) is useful for evaluating limits of functions in two variables because these paths allow us to explore how the function behaves along different directions approaching the origin. By adjusting the exponents α and β, we can test a variety of trajectories that the function might take, revealing whether the limit depends on the direction of approach.

Example 2.15. Let $f : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ be defined by

$$f(x,y) = \frac{x^3y^3}{x^4+y^{12}}.$$

Our goal is to determine the limit

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

First, let us test all linear paths by considering

$$\lim_{t\to 0} f(\alpha t, \beta t),$$

with $\alpha, \beta \in \mathbb{R}$ not both zero. In this case, we get

$$\lim_{t\to 0} f(\alpha t,\beta t) = \lim_{t\to 0} \frac{\alpha^3\beta^3t^6}{\alpha^4t^4+\beta^{12}t^{12}} = \lim_{t\to 0} \frac{\alpha^3\beta^3t^2}{\alpha^4+\beta^{12}t^8} = 0.$$

We see that all linear paths yield the same limit. Therefore, to demonstrate that the limit does not exist, we must consider non-linear paths.

When dealing with a denominator containing different powers of x and y, a good approach is to examine paths of the form (t^α, t^β) for various values of $\alpha, \beta \in (0,\infty)$. This gives

$$\lim_{t\to 0} f(t^\alpha, t^\beta) = \lim_{t\to 0} \frac{t^{3\alpha+3\beta}}{t^{4\alpha}+t^{12\beta}}.$$

First, we can take $\alpha=\beta=1$. In this case we have

$$\lim_{t\to 0} f(t,t) = \lim_{t\to 0} \frac{t^6}{t^4+t^{12}} = \lim_{t\to 0} \frac{t^2}{1+t^8} = 0.$$

Next, we choose α and β so that the powers appearing in the denominator match. For us, this means we want to find α and β such that

$$4\alpha = 12\beta.$$

For example, this is achieved by taking $\alpha = 3$ and $\beta = 1$. Then,

$$\lim_{t\to 0} f(t^3, t) = \lim_{t\to 0} \frac{t^{12}}{t^{12} + t^{12}} = \frac{1}{2}.$$

Since $\alpha = \beta = 1$ and $\alpha = 3, \beta = 1$ yield different results, we conclude that the limit does not exist.

2.5 Continuity at a Point

The purpose of this section is to introduce and discuss continuous functions in several variables.

Definition 2.5 (Continuous function at a point)**.** Let $E \subseteq \mathbb{R}^n$ and let $\mathbf{x}_0$ be an interior point of E. A function $f\colon E \to \mathbb{R}$ is said to be continuous at $\mathbf{x}_0$ if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0).$$

Definition 2.6 (1^{st} equivalent definition)**.** Let $\mathbf{x}_0$ be an interior point of E. A function $f\colon E \to \mathbb{R}$ is continuous at $\mathbf{x}_0$ if and only if, for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for all $\mathbf{x} \in E$,

$$d(\mathbf{x}, \mathbf{x}_0) \leqslant \delta \implies |f(\mathbf{x}) - f(\mathbf{x}_0)| \leqslant \varepsilon.$$

Definition 2.7 (2^{nd} equivalent definition)**.** Let $\mathbf{x}_0$ be an interior point of E. A function $f\colon E \to \mathbb{R}$ is continuous at $\mathbf{x}_0$ if and only if, for every sequence $(\mathbf{a}_k)_{k\in\mathbb{N}}$ of elements of E we have

$$\lim_{k\to+\infty} \mathbf{a}_k = \mathbf{x}_0 \implies \lim_{k\to+\infty} f(\mathbf{a}_k) = f(\mathbf{x}_0).$$

Remark 2.1. It is very tempting to believe that if a function is continuous in every coordinate then the function is continuous. However, this is NOT TRUE! As a counterexample, consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

Let $f_1, f_2\colon \mathbb{R} \to \mathbb{R}$ denote the two functions obtained by restricting $f(x,y)$ to the first and second coordinate at the point $(0,0)$, that is, $f_1(x) = f(x,0)$ and $f_2(y) = f(0,y)$. Then $f_1(x)$ and $f_2(y)$ both are continuous at $x = 0$ and $y = 0$ respectively. Nonetheless, we have already seen in Example 2.9 that the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ does not exist, which means that the function $f(x,y)$ (as a function in two variables) is not continuous at the point $(0,0)$.

Properties of continuity. Let f and g be two functions from $E \subseteq \mathbb{R}^n$ to $\mathbb{R}$ that are

continuous at a point $\mathbf{x}_0 \in \mathbb{R}^n$. Then:
1. **Linearity**: For any $\alpha, \beta \in \mathbb{R}$, the function $\alpha f + \beta g$ is continuous at $\mathbf{x}_0$;
2. **Products**: The product function fg is continuous at $\mathbf{x}_0$;
3. **Quotients**: If $g(\mathbf{x}_0) \neq 0$ and $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in E$ then the quotient $\frac{f}{g}$ is continuous at $\mathbf{x}_0$;
4. **Compositions**: Let A be a subset of $\mathbb{R}^n$ and let

$$g_1, \ldots, g_p : A \to \mathbb{R}$$

be functions continuous at the point $\mathbf{a} = (a_1, \ldots, a_n)$. On the other hand, let B be a subset of $\mathbb{R}^p$ containing

$$\{(g_1(\mathbf{y}), \ldots, g_p(\mathbf{y})) : \mathbf{y} \in A\}$$

and $f\colon B \to \mathbb{R}$ a function continuous at the point $\mathbf{b} = (g_1(\mathbf{a}), \ldots, g_p(\mathbf{a}))$. Then the function $F\colon A \to \mathbb{R}$ defined by

$$F(y_1, \ldots, y_n) = f(g_1(y_1, \ldots, y_n), \ldots, g_p(y_1, \ldots, y_n))$$

is continuous at the point $\mathbf{a} = (a_1, \ldots, a_n)$.

Example 2.16. Let us demonstrate the usefulness of the properties of continuity by showing that the function $F\colon \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = -\sin(x)y$ is continuous at the point $(0, 0)$. To do this, consider the three auxiliary functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g_1, g_2\colon \mathbb{R} \to \mathbb{R}$ defined respectively by

$$f(x, y) = xy, \qquad g_1(x, y) = -\sin(x), \qquad \text{and} \qquad g_2(x, y) = y.$$

Since both $g_1(x, y)$ and $g_2(x, y)$ are continuous at $(0, 0)$ and $f(x, y)$ is continuous at $(g_1(0, 0), g_2(0, 0)) = (0, 0)$, we can conclude that $F(x, y) = f(g_1(x, y), g_2(x, y))$ is continuous at the point $(0, 0)$ (See Fig. 2.8).

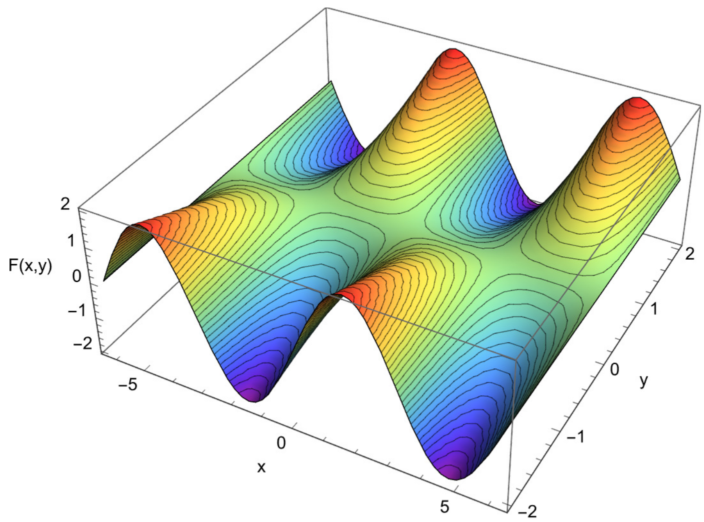


Figure 2.8: Graph of the function $F(x, y) = -\sin(x)y$.

2.6 Continuity in a Region

Definition 2.8 (Continuous function in a Region)**.** Let E be a non-empty subset of $\mathbb{R}^n$. A function $f\colon E \to \mathbb{R}$ is continuous on E if for every $\mathbf{x}_0 \in E$ and every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that for all $\mathbf{x} \in E$,

$$d(\mathbf{x}, \mathbf{x}_0) \leqslant \delta \implies |f(\mathbf{x}) - f(\mathbf{x}_0)| \leqslant \varepsilon.$$

Definition 2.9 (Equivalent definition)**.** Let E be a non-empty subset of $\mathbb{R}^n$. A function $f\colon E \to \mathbb{R}$ is continuous on E if for every sequence $(\mathbf{a}_k)_{k\in\mathbb{N}}$ of elements of E we have

$$\lim_{k\to+\infty} \mathbf{a}_k = \mathbf{x}_0 \implies \lim_{k\to+\infty} f(\mathbf{a}_k) = f(\mathbf{x}_0).$$

Remark 2.2. If E is an open set then $f\colon E \to \mathbb{R}$ is continuous on E if and only if it is continuous at every point in E.

Example 2.17. Let us demonstrate that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{\sin(xy)}{x} & \text{if } x \neq 0 \\ y & \text{if } x = 0 \end{cases}$$

is continuous on $\mathbb{R}^2$ (see Fig. 2.9).

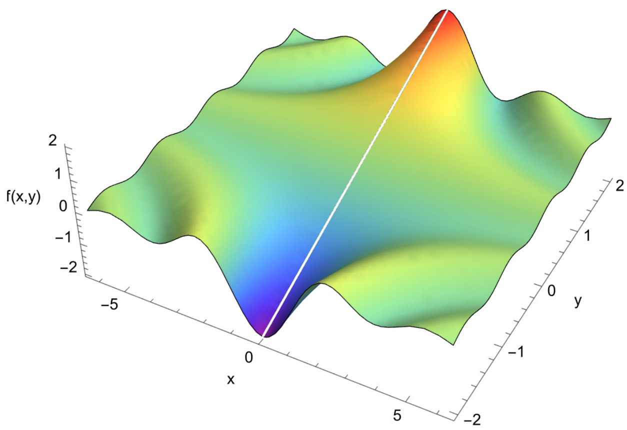


Figure 2.9: Graph of $f(x,y) = \frac{\sin(xy)}{x}$ for $x \neq 0$.

Define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(s) = \begin{cases} \frac{\sin(s)}{s} & \text{if } s \neq 0 \\ 1 & \text{if } s = 0 \end{cases}$$

It is continuous for all $s \neq 0$ and, as $\lim_{s\to 0} h(s) = 1 = h(0)$, it is also continuous at 0. This is useful because we have $f(x,y) = h(xy)y$ for all $(x,y) \in \mathbb{R}^2$. Since the functions

$$a(x,y) = xy \qquad \text{and} \qquad b(x,y) = y$$

are continuous at every point in $\mathbb{R}^2$ and $f(x,y) = h(xy)y = a(h(a(x,y)), b(x,y))$ for all $(x,y) \in \mathbb{R}^2$, it follows from the properties of continuity that f is continuous at every point in $\mathbb{R}^2$.

2.7 Extreme Value Theorem and Intermediate Value Theorem

Definition 2.10 (Maximum and minimum)**.** Let $E \subseteq \mathbb{R}^n$ be non-empty and f a function from E to $\mathbb{R}$. A real number M satisfying

- $f(\mathbf{x}) \leqslant M$ for every element $\mathbf{x}$ in E, and
- $M \in \operatorname{Im}(f)$,

is called the *maximum* of the function f on E and is denoted by $\max_{\mathbf{x} \in E} f(\mathbf{x})$. If $\mathbf{x}_0 \in E$ is such that $f(\mathbf{x}_0) = M$ then we say that the function f reaches its maximum at the point $\mathbf{x}_0$. Similarly, a real number m satisfying

- $f(\mathbf{x}) \geqslant m$ for every element $\mathbf{x}$ in E, and
- $m \in \operatorname{Im}(f)$,

is called the *minimum* of the function f on E and is denoted by $\min_{\mathbf{x} \in E} f(\mathbf{x})$. If $\mathbf{x}_0 \in E$ is such that $f(\mathbf{x}_0) = M$ then we say that the function f reaches its minimum at the point $\mathbf{x}_0$.

Proposition 2.3 (Extreme value theorem)**.** *Let E be a compact subset of $\mathbb{R}^n$ and $f\colon E \to \mathbb{R}$ a continuous function. Then f has a minimum $\min_{\mathbf{x} \in E} f(\mathbf{x})$ and a maximum $\max_{\mathbf{x} \in E} f(\mathbf{x})$ on E.*

Chapter 3

Partial derivatives and differentiability

3.1 Partial Derivatives

Recall that given a differentiable function in a single variable $f\colon \mathbb{R} \to \mathbb{R}$, the derivative of f at the point $a \in \mathbb{R}$ is defined as

$$f'(a) = \frac{df}{dx}(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We are already familiar with several different ways of thinking about the derivative of a function:

- The derivative of a function f quantifies the rate of change of the function's output value with respect to its input value. For example, if the derivative of f at a point a is a 'large' positive number then a positive change close to a will result in a 'proportionately large' positive change in the output value. Conversely, if the derivative of f at a point a is a 'small' negative number then a positive change close to a will result in a 'proportionately small' negative change in the output value.
- The derivative $f'(a)$ of a function f at a point a equals the slope of the tangent line to the graph of the function at that point. Moreover, the tangent line is the best linear approximation of the function near that input value.

The goal of this chapter is to extend derivatives to functions in several variables. While functions in one variable have only one derivative, functions in several variables have multiple derivatives, one for each variable. These are called the partial derivatives.

Let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots, \ \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

denote the vectors of the canonical basis of $\mathbb{R}^n$. Note that for any element $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we have $\mathbf{x} = \sum_{k=1}^n x_k \mathbf{e}_k$, where $x_k = \langle \mathbf{x}, \mathbf{e}_k \rangle$, for $k = 1, \ldots, n$.

Definition 3.1 (Partial derivatives)**.** Suppose $E \subseteq \mathbb{R}^n$ is a set and $\mathbf{a} = (a_1, \ldots, a_n)$ is an interior point of E. Let $f \colon E \to \mathbb{R}$ be a real-valued function in the variables $(x_1, \ldots, x_n)$. The *partial derivative* of f at the point $\mathbf{a}$ with respect to the variable x_k (the k-th variable) is defined as

$$\frac{\partial f}{\partial x_k}(\mathbf{a}) = \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{e}_k) - f(\mathbf{a})}{t}$$

whenever this limit exists. If this limit does not exist then we say that the partial derivative of f at $\mathbf{a}$ with respect to x_k does not exist.

Intuitively, the partial derivative $\frac{\partial f}{\partial x_k}$ is the derivative of $f(x_1, \ldots, x_n)$ with respect to the variable x_k while all the other variables remain constant. We also use the notation

$$D_k f(\mathbf{a}) = \frac{\partial f}{\partial x_k}(\mathbf{a});$$

or if the real variables of f are explicitly given, say $f(x, y, z)$, then we write

$$\begin{aligned} D_x f(x, y, z) &= \frac{\partial f}{\partial x}(x, y, z) \\ D_y f(x, y, z) &= \frac{\partial f}{\partial y}(x, y, z) \\ D_z f(x, y, z) &= \frac{\partial f}{\partial z}(x, y, z). \end{aligned}$$

Remark 3.1. The partial derivative $\frac{\partial f}{\partial x_k}(\mathbf{a})$ exists if and only if the function $g_k(t) = f(\mathbf{a} + t\mathbf{e}_k)$ is differentiable at $t = 0$, because

$$\frac{\partial f}{\partial x_k}(\mathbf{a}) = \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{e}_k) - f(\mathbf{a})}{t} = \lim_{t \to 0} \frac{g_k(t) - g_k(0)}{t} = g_k'(0). \tag{3.1}$$

This means that $\frac{\partial f}{\partial x_k}(\mathbf{a})$ corresponds to the slope of the tangent line pointing in the direction of the canonical vector $\mathbf{e}_k$. In the case of two variables, Fig. 3.1 below provides an illustration of the partial derivatives as the slope of tangent lines in the x-direction and in the y-direction.

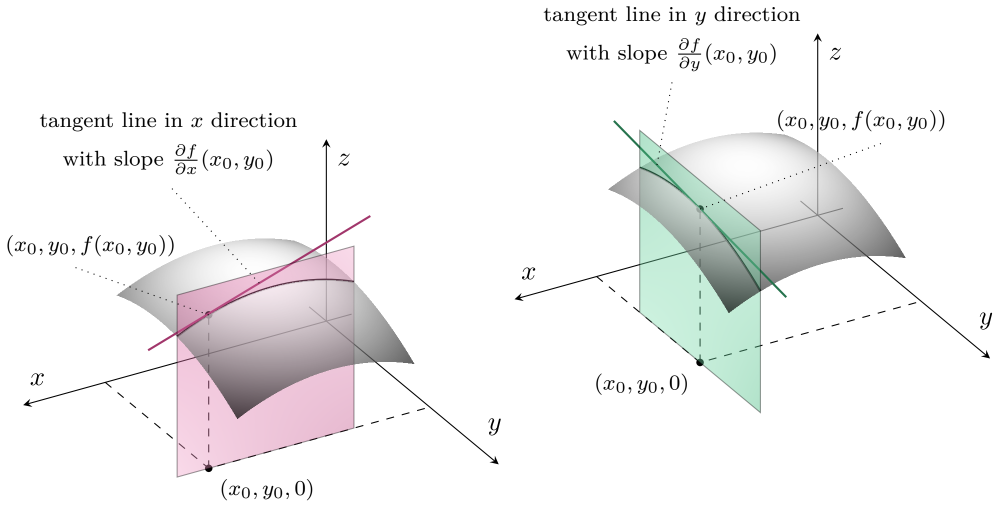


Figure 3.1: The gray surface is the graph of the function $f(x, y)$ and contains the point $(x_0, y_0, f(x_0, y_0))$. In the left figure, the plane $y = y_0$ (pink plane) intersects the graph of $f(x, y)$ in a curve. The tangent line to this curve at the point $(x_0, y_0, f(x_0, y_0))$ (pink line) has slope equal to the partial derivative of $f(x, y)$ with respect to the variable x at the point (x_0, y_0). The right figure depicts the tangent line (green line) to the curve that is the intersection of the graph of $f(x, y)$ with the plane $x = x_0$ (green plane) at the point $(x_0, y_0, f(x_0, y_0))$, whose slope is the partial derivative of $f(x, y)$ with respect to the variable y at the point (x_0, y_0).

Example 3.1. Consider a pot filled with water being heated on top of a stove (see Fig. 3.2). Let us think of the pot as a cylinder in $\mathbb{R}^3$ given by

$$D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1,\ 0 < z < 1\}.$$

Suppose at time t the temperature of the water at the position (x, y, z) is given by the equation

$$T(x, y, z, t) = \left(100 - \frac{80}{1+t}\right) \cdot \left(1 - \frac{z}{2}\right) \cdot e^{-x^2-y^2}.$$

Then T is a function in 4 variables (3 space variables and 1 time variable) with domain $\operatorname{dom}(T) = D \times [0, \infty)$. We can calculate its partial derivatives as

$$\begin{aligned}
D_x T(x, y, z, t) &= \frac{\partial T}{\partial x}(x, y, z, t) = \left(100 - \frac{80}{1+t}\right) \cdot \left(1 - \frac{z}{2}\right) \cdot (-2x) \cdot e^{-x^2-y^2}, \\
D_y T(x, y, z, t) &= \frac{\partial T}{\partial y}(x, y, z, t) = \left(100 - \frac{80}{1+t}\right) \cdot \left(1 - \frac{z}{2}\right) \cdot (-2y) \cdot e^{-x^2-y^2}, \\
D_z T(x, y, z, t) &= \frac{\partial T}{\partial z}(x, y, z, t) = \left(100 - \frac{80}{1+t}\right) \cdot \left(-\frac{1}{2}\right) \cdot e^{-x^2-y^2}, \\
D_t T(x, y, z, t) &= \frac{\partial T}{\partial t}(x, y, z, t) = \frac{80}{(1+t)^2} \cdot \left(1 - \frac{z}{2}\right) \cdot e^{-x^2-y^2}.
\end{aligned}$$

What do these partial derivatives mean? For example, $T_t(x, y, z, t)$ describes the rate of change in temperature at a stationary point (x, y, z) as time t changes. Since T_t is always positive, we see that in every point (x, y, z) the temperature is increasing as the time t increases. Conversely, due to the sign of T_x, T_y, T_z, we see that for a fixed time t, the temperature is decreasing as we move away from the origin and towards the boundary of the cylinder, which makes sense because the water at the edge of the pot should be cooler than the water in the middle.

Figure 3.2: A pot of water with heat being applied from the bottom.

Definition 3.2 (Gradient vector)**.** Let $E \subseteq \mathbb{R}^n$ be an open set, let $f: E \to \mathbb{R}$ be a function and suppose all partial derivatives $\frac{\partial f}{\partial x_1}(\mathbf{a}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{a})$ of f at the point $\mathbf{a} \in E$ exist. Then

$$\nabla f(\mathbf{a}) = \operatorname{grad} f(\mathbf{a}) := \left(\frac{\partial f}{\partial x_n}(\mathbf{a}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{a}) \right) \in \mathbb{R}^{1 \times n},$$

is called *the gradient of* f *at* $\mathbf{a}$. If at least one of the partial derivatives $\frac{\partial f}{\partial x_1}(\mathbf{a}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{a})$ of f at the point $\mathbf{a}$ does not exist then we say that the gradient of f at $\mathbf{a}$ does not exist.

Remark 3.2. The gradient $\nabla f(\mathbf{a})$ can also be written as a linear combination using the canonical vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$,

$$\nabla f(\mathbf{a}) = \sum_{k=1}^{n} D_k f(\mathbf{a}) \mathbf{e}_k^\top.$$

Therefore $D_k f(\mathbf{a}) = \frac{\partial f}{\partial x_k}(\mathbf{a}) = \langle \nabla f(\mathbf{a}), \mathbf{e}_k \rangle$ for all $k = 1, 2, \dots, n$.

3.2 Directional Derivatives

Definition 3.3 (Directional derivatives)**.** Let $E \subseteq \mathbb{R}^n$ be an open set, $f: E \to \mathbb{R}$ a real-valued function, and $\mathbf{v} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$. The *directional derivative* of f along the vector

$\mathbf{v}$ at the point $\mathbf{a} \in E$ is defined as

$$\nabla_{\mathbf{v}} f(\mathbf{a}) = \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t}$$

wherever this limit exists. If this limit does not exist then we say that the directional derivative of f along $\mathbf{v}$ at the point $\mathbf{a}$ does not exist. When $\mathbf{v}$ is a unit vector (which means $\|\mathbf{v}\|_2 = 1$), it is also called the *derivative in the direction* $\mathbf{v}$.

Note that the partial derivative with respect to the variable x_k coincides with the directional derivative along the vector $\mathbf{e}_k$, that is,

$$\frac{\partial f}{\partial x_k}(\mathbf{a}) = \nabla_{\mathbf{e}_k} f(\mathbf{a}).$$

Properties of directional/partial derivatives: Many of the familiar properties of the ordinary derivative hold for the directional derivative. In particular, if $\nabla_{\mathbf{v}} f(\mathbf{a})$ and $\nabla_{\mathbf{v}} g(\mathbf{a})$ exist then

1. **Linearity**: For all $\alpha, \beta \in \mathbb{R}$ we have

$$\nabla_{\mathbf{v}}(\alpha f + \beta g)(\mathbf{a}) \;=\; \alpha(\nabla_{\mathbf{v}} f(\mathbf{a})) + \beta(\nabla_{\mathbf{v}} g(\mathbf{a})).$$

2. **Product rule** (or **Leibniz's rule**):

$$\nabla_{\mathbf{v}}(f \cdot g)(\mathbf{a}) \;=\; g(\mathbf{a}) \cdot \nabla_{\mathbf{v}} f(\mathbf{a}) \;+\; f(\mathbf{a}) \cdot \nabla_{\mathbf{v}} g(\mathbf{a}).$$

3. **Quotient rule**: If $g(\mathbf{a}) \neq 0$ then

$$\nabla_{\mathbf{v}}\Big(\frac{f}{g}\Big)(\mathbf{a}) \;=\; \frac{g(\mathbf{a}) \cdot \nabla_{\mathbf{v}} f(\mathbf{a}) \;-\; f(\mathbf{a}) \cdot \nabla_{\mathbf{v}} g(\mathbf{a})}{g(\mathbf{a})^2}.$$

3.3 Differentiability at a Point

Recall from linear algebra that a *linear map* from $\mathbb{R}^n$ to $\mathbb{R}$ is a function $L\colon \mathbb{R}^n \to \mathbb{R}$ that satisfies linearity, meaning it preserves addition and scalar multiplication: for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$, we have

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha L(\mathbf{x}) + \beta L(\mathbf{y}).$$

Note that any linear map L can always be represented as $L(\mathbf{x}) = \langle \mathbf{w}, \mathbf{x} \rangle$, where $\mathbf{w} \in \mathbb{R}^n$ is a fixed vector and $\langle ., . \rangle$ denotes the standard inner product on $\mathbb{R}^n$ defined in (1.1).

Definition 3.4 (Differentiability at a point)**.** Let E be a non-empty open subset of $\mathbb{R}^n$. A function $f\colon E \to \mathbb{R}$ is *differentiable* at the point $\mathbf{a} \in E$ if there exists a linear map $L_{\mathbf{a}}\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{\big|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - L_{\mathbf{a}}(\mathbf{h})\big|}{\|\mathbf{h}\|_2} = 0.$$

In this case, the linear map $L_{\mathbf{a}}\colon \mathbb{R}^n \to \mathbb{R}$ is called the **differential** of f at the point $\mathbf{a}$.

Theorem 3.1 (Fundamental theorem)**.** *Suppose $f\colon E \to \mathbb{R}$ is a function defined on a set $E \subseteq \mathbb{R}^n$, and $\mathbf{a}$ is an interior point of E. If f is differentiable at $\mathbf{a}$ then the following statements hold:*

(i) *f is continuous at $\mathbf{a}$.*

(ii) *All partial derivatives of f at the point $\mathbf{a}$ exist, the gradient vector $\nabla f(\mathbf{a})$ of f at the point $\mathbf{a}$ exists, and the differential $L_{\mathbf{a}}\colon \mathbb{R}^n \to \mathbb{R}$ of f at the point $\mathbf{a}$ is the same as scalar multiplication by the gradient vector, i.e.,*

$$L_{\mathbf{a}}(\mathbf{v}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}, \qquad \forall \mathbf{v} \in \mathbb{R}^n.$$

(iii) *All directional derivatives of f at the point $\mathbf{a}$ exist and are given by*

$$\nabla_{\mathbf{v}} f(\mathbf{a}) = L_{\mathbf{a}}(\mathbf{v}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}, \qquad \forall \mathbf{v} \in \mathbb{R}^n.$$

(iv) *For all $\mathbf{x} \in E$ we have*

$$f(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + r_1(\mathbf{x}),$$

where r_1 is an "error" term satisfying

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{r_1(\mathbf{x})}{\|\mathbf{x} - \mathbf{a}\|_2} = 0.$$

The function

$$t(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

*is called the **linearization** (or **linear approximation**) of f at the point $\mathbf{a}$.*

(v) *The function $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ increases most rapidly in the direction ∇f, and decreases most rapidly in the direction $-\nabla f$. Any vector $\mathbf{v} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ orthogonal to ∇f is a direction of zero change.*

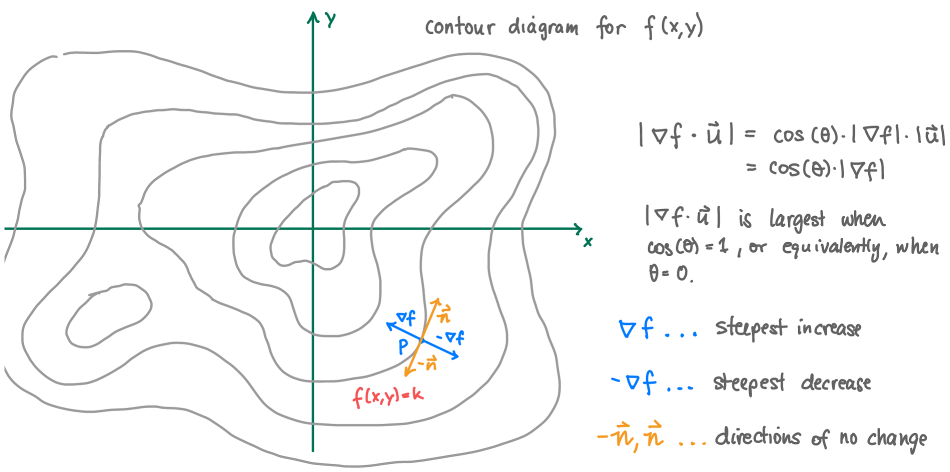


Figure 3.3: The gradient vector ∇f gives the direction of steepest incline, while the rate of change in the direction of the contour lines equals 0.

Remark 3.3. The gradient is perpendicular to the level sets of a function.

Theorem 3.2 (Sufficient conditions for differentiability)**.** *Let $E \subseteq \mathbb{R}^n$, $f\colon E \to \mathbb{R}$, and suppose* $\mathbf{a}$ *is an interior point of E. If there exists $\delta > 0$ such that every partial derivative $\frac{\partial f}{\partial x_k}$ of f exists at every point in the open ball $B(\mathbf{a}, \delta)$ and $\frac{\partial f}{\partial x_k}(x_1, \ldots, x_k)$ is a continuous function at the point* $\mathbf{a}$*, then f is differentiable at the point* $\mathbf{a}$*.*

Example 3.2. Consider $n = 2, E = \mathbb{R}^2, f\colon \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x^2 - y^2$. We have:

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= 2x, \\ \frac{\partial f}{\partial y}(x, y) &= -2y, \\ \nabla f(x, y) &= (2x, -2y). \end{aligned}$$

Example 3.3. Let $E = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and $f(x, y) = e^{y \log x}$. Then

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \frac{y e^{y \log x}}{x}, \\ \frac{\partial f}{\partial y}(x, y) &= e^{y \log x} \cdot \log x, \\ \nabla f(x, y) &= \left(\frac{y e^{y \log x}}{x}, e^{y \log x} \cdot \log x \right). \end{aligned}$$

3.4 Tangent (Hyper)Planes

Recall that a straight line is called a *tangent line* to the curve $y = f(x)$ at a point $x = a$ if the line passes through the point $(a, f(a))$ on the curve and has slope $f'(a)$, where $f'(x)$ is the 1^{st} derivative of f. The equation of the tangent line is then given by

$$y = f(a) + f'(a)(x - a).$$

The equation of the tangent line is closely related to Taylor's theorem, which says that the 1^{st}-order Taylor expansion of f is given by

$$f(x) = \underbrace{\overbrace{f(a) + f'(a)(x - a) +}^{\textit{linear approximation}} \overbrace{r_1(x)}^{\textit{remainder}}}_{1^{\text{st}}\textit{-order expansion}}$$

where $r_1(x)$ is an "error" term that satisfies $\lim_{x \to a} \frac{r_1(x)}{|x-a|} = 0$.

A similar concept applies to multivariate functions in n-dimensional Euclidean space. As we have seen (cf. part (iv) of Theorem 3.1) if $f(x_1, \ldots, x_n)$ is a function in

n variables that is differentiable at a point $\mathbf{a} \in \mathbb{R}^n$ then

$$f(\mathbf{x}) = f(\mathbf{a}) + L_{\mathbf{a}}(\mathbf{x}-\mathbf{a}) + r_1(\mathbf{x}) = \underbrace{\overbrace{f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a})}^{\textit{linear approximation}} + \overbrace{r_1(\mathbf{x})}^{\textit{remainder}}}_{1^{\text{st}}\textit{-order expansion}} \tag{3.2}$$

where $r_1(\mathbf{x})$ is an "error" term satisfying $\lim_{\mathbf{x}\to\mathbf{a}} \frac{r_1(\mathbf{x})}{\|\mathbf{x}-\mathbf{a}\|_2} = 0$.

Definition 3.5 (Tangent hyperplane)**.** Let $E \subseteq \mathbb{R}^n$ and $f\colon E \to \mathbb{R}$, and assume that $\mathbf{a}$ is an interior point of E. Suppose f is differentiable at $\mathbf{a}$, and consider the linear approximation of f at $\mathbf{a}$ given by

$$t(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a}).$$

The graph of $t(\mathbf{x})$ is called the *tangent hyperplane* of f at $\mathbf{a}$. That is, the tangent hyperplane consists of all points $(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$ satisfying the equation

$$x_{n+1} = t(x_1, \ldots, x_n).$$

This equation is commonly referred to as the *equation of the tangent hyperplane.*

When $n = 1$, the tangent hyperplane is the same as the tangent line, and when $n = 2$ the tangent hyperplane is usually just called the *tangent plane* (see Fig. 3.4).

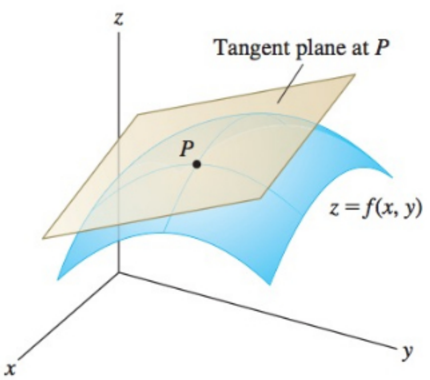


Figure 3.4: Tangent plane to a function $z = f(x,y)$ at $P = (x_0, y_0, f(x_0, y_0))$.

Example 3.4. Let us find the equation of the tangent plane to the elliptic paraboloid

$$z = 2x^2 + y^2 + 1$$

at the point $(1, -1, 4)$. This elliptic paraboloid is the graph of the function $f(x,y) = 2x^2 + y^2 + 1$. The partial derivatives of f form the gradient given by

$$\nabla f(x,y) = (4x, 2y).$$

We can now write down the linear approximation of $f(x, y)$ at the point $(1, -1)$ as

$$\begin{aligned} t(x,y) &= f(1,-1) + \nabla f(1,-1) \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \\ &= 4 + (4, -2) \cdot \begin{pmatrix} x-1 \\ y+1 \end{pmatrix} \\ &= 4 + 4(x-1) - 2(y+1) \\ &= 4x - 2y - 2. \end{aligned}$$

Thus, the equation of the tangent plane to the elliptic paraboloid at the point $(1, -1, 4)$ is

$$z = 4x - 2y - 2.$$

3.5 Functions of Class C^1

Definition 3.6 (Differentiability in a region)**.** Let $E \subseteq \mathbb{R}^n$ be an open set and $f \colon E \to \mathbb{R}$ a function on E. If f is differentiable at every point $\mathbf{a} \in E$ then we say that f is *differentiable on* E.

Definition 3.7 (Functions of Class C^1)**.** Let $E \subseteq \mathbb{R}^n$ be an open set. A function $f \colon E \to \mathbb{R}$ is said to be *of class* $C^1(E)$ if all its partial derivatives exist and are continuous at each point $\mathbf{x} \in E$.

The existence and continuity of the partial derivatives at every point in E implies the differentiability of the function at every point in E (see Theorem 3.2). It follows that any function of class $C^1(E)$ is differentiable on E.

Proposition 3.1. *Let $E \subseteq \mathbb{R}^n$ be open and $f \colon E \to \mathbb{R}$ a function of class $C^1(E)$. Then f is differentiable on E.*

Example 3.5. Consider the function $f \colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

We have already studied this function in Example 2.9 and Remark 2.1.

- For $(x, y) \neq (0, 0)$, we can calculate the partial derivatives as

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \frac{y}{x^2+y^2} - \frac{2x^2y}{(x^2+y^2)^2} \\ \frac{\partial f}{\partial y}(x,y) &= \frac{x}{x^2+y^2} - \frac{2xy^2}{(x^2+y^2)^2}. \end{aligned}$$

- At the point $(0,0)$ we can use the definition of partial derivatives and find

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

This shows that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist for every point in $\mathbb{R}^2$. Nonetheless, this function is **not** differentiable at the point $(0,0)$. Indeed, we have seen in Example 2.9 that this function is not even continuous at the point $(0,0)$, so according to part (i) of Theorem 3.1, it cannot be differentiable at that point. This example illustrates that even if a function is differentiable in every coordinate, this does not mean that it is differentiable. In conclusion, the function f is of class $C^1(\mathbb{R}^2\setminus\{(0,0)\})$.

3.6 Second Order Partial Derivatives

The partial derivatives $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ are also referred to as *"partial derivatives of order 1"* or *"first order partial derivatives"*. Let us now define the second order partial derivatives.

Definition 3.8 (Partial derivatives of second order)**.** Let $E \subseteq \mathbb{R}^n$ be an open set and $1 \leqslant k \leqslant n$. Assume $f\colon E \to \mathbb{R}$ is a function whose partial derivative $\frac{\partial f}{\partial x_k}$ exists for every point in E. For $1 \leqslant i \leqslant n$, if the partial derivative of $\frac{\partial f}{\partial x_k}$ with respect to the variable x_i at the point a exists, then we obtain a *second order partial derivative* of f with respect to x_i and x_k at $\mathbf{a}$ denoted by $\frac{\partial^2 f}{\partial x_i \partial x_k}(a)$. If this derivative exists for every $\mathbf{a} \in E$, it defines a function $\frac{\partial^2 f}{\partial x_i \partial x_k} : E \to \mathbb{R}$.

If $i = k$, then it is also common to write $\frac{\partial^2 f}{\partial x_i^2}$ instead of $\frac{\partial^2 f}{\partial x_i \partial x_i}$. If $i \neq k$, then there are generally two mixed second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x_i \partial x_k} \quad \text{and} \quad \frac{\partial^2 f}{\partial x_k \partial x_i}.$$

These derivatives are not necessarily equal since the order of differentiation can affect the result. However, as the following theorem states, they are equal if an additional continuity assumption is satisfied.

Theorem 3.3 (Schwarz's theorem)**.** *Let $E \subseteq \mathbb{R}^n$ be open and let $f\colon E \to \mathbb{R}$ be a function defined on E. For any point $\mathbf{a} \in E$ and indices $i, k \in \{1, \dots, n\}$, suppose the mixed partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_k}$ and $\frac{\partial^2 f}{\partial x_k \partial x_i}$ exist in E and are continuous at $\mathbf{a}$. Then, $\frac{\partial^2 f}{\partial x_i \partial x_k}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{a})$.*

Definition 3.9. The $n \times n$ matrix

$$\mathrm{Hess}(f)(\mathbf{a}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{a}) \end{pmatrix}$$

is called the *Hessian matrix* of f at the point $\mathbf{a}$, written $\mathrm{Hess}(f)(\mathbf{a})$.

If all the partial derivatives of order 2 exist and are continuous at $\mathbf{a}$ then by Schwarz's theorem the Hessian matrix is a symmetric matrix, i.e., $\mathrm{Hess}(f)(\mathbf{a}) = \mathrm{Hess}(f)(\mathbf{a})^T$. In this case we can use the Hessian matrix to form the *second order expansion* of a differentiable function, given by

$$f(\mathbf{x}) = \underbrace{\overbrace{\overbrace{f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a})}^{\textit{linear approximation}} + \frac{1}{2}(\mathbf{x}-\mathbf{a})^T \cdot \mathrm{Hess}(f)(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a})}^{\textit{quadratic approximation}} + \overbrace{r_2(\mathbf{x})}^{\textit{remainder}}}_{2^{\text{nd}}\textit{-order expansion}} \tag{3.3}$$

where $r_2(\mathbf{x})$ is an "error" term satisfying $\lim_{\mathbf{x}\to\mathbf{a}} \frac{r_2(\mathbf{x})}{\|\mathbf{x}-\mathbf{a}\|_2^2} = 0$.

The quadratic approximation is a polynomial of degree 2 in n variables called the *Taylor polynomial of order* 2 *at the point* $\mathbf{a}$ and it is usually denoted by $P_2(x, y)$.

Example 3.6. Let us find the Taylor polynomial of order 2 for the function $f(x, y) = \sin(2x + y) + 3\cos(x + y)$ at the point $(0, 0)$. Recall the formula for computing the quadratic approximation of a function in two variables at the point $(0, 0)$ is

$$P_2(x, y) = f(0, 0) + \nabla f(0, 0) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x, y) \cdot \ \mathrm{Hess}\ (f)(0, 0) \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

To use this formula, we have to find the gradient vector and the Hessian matrix first. We have

$$\nabla f(x, y) = (2\cos(2x + y) - 3\sin(x + y), \cos(2x + y) - 3\sin(x + y))$$

which gives

$$\nabla f(0, 0) = (2, 1).$$

Moreover,

$$\mathrm{Hess}(f)(x, y) = \begin{pmatrix} -4\sin(2x + y) - 3\cos(x + y) & -2\sin(2x + y) - 3\cos(x + y) \\ -2\sin(2x + y) - 3\cos(x + y) & -\sin(2x + y) - 3\cos(x + y) \end{pmatrix}$$

and hence

$$\mathrm{Hess}(f)(0, 0) = \begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}.$$

It follows that

$$\begin{aligned}P_2(x,y) &= 3 + (2,1)\cdot\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x,y)\cdot\begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} \\ &= 3 + 2x + y - \frac{3}{2}x^2 - 3xy - \frac{3}{2}y^2.\end{aligned}$$

This is a degree 2 polynomial in 2 variables.

3.7 Higher Order Partial Derivatives

Definition 3.10 (Partial derivatives of higher orders)**.** Consider a function $f\colon E \to \mathbb{R}$ defined on an open set $E \subseteq \mathbb{R}^n$. For a sequence of indices $i_1, \ldots, i_p$ with each $i_j \in \{1, \ldots, n\}$ and for $p \geqslant 3$, assume that the $(p-1)$-th order partial derivative of f, denoted as $\frac{\partial^{p-1} f}{\partial x_{i_1} \ldots \partial x_{i_{p-1}}}$, exists in E. Then, the p-th order partial derivative of f with respect to these indices, if it exists, is given by:

$$\frac{\partial^p f}{\partial x_{i_p} \ldots \partial x_{i_1}} = \frac{\partial}{\partial x_{i_p}}\left(\frac{\partial^{p-1} f}{\partial x_{i_1} \ldots \partial x_{i_{p-1}}}\right).$$

This derivative is denoted as $\frac{\partial f}{\partial x_{i_p} \ldots \partial x_{i_1}}(\mathbf{a})$ for any point $\mathbf{a} \in E$. If such a derivative exists for every $\mathbf{a} \in E$, it defines a function $\frac{\partial^p f}{\partial x_{i_p} \ldots \partial x_{i_1}} : E \to \mathbb{R}$.

Example 3.7. Consider a function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x,y) = x^3y^2$. We calculate its higher-order partial derivatives as follows:

$$\begin{aligned}&\frac{\partial f}{\partial x}(x,y) = 3x^2y^2, \\ &\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}(3x^2y^2) = 6xy^2, \\ &\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}(3x^2y^2) = 6x^2y, \\ &\frac{\partial^3 f}{\partial y \partial x^2}(x,y) = \frac{\partial}{\partial y}(6xy^2) = 12xy, \\ &\frac{\partial^3 f}{\partial x^3}(x,y) = \frac{\partial}{\partial x}(6xy^2) = 6y^2.\end{aligned}$$

This illustrates the computation of first, second, and third-order partial derivatives for a function of two variables.

Remark 3.4. Explicit computations also give $\frac{\partial^2 f}{\partial x \partial y}(x,y) = 6x^2y = \frac{\partial^2 f}{\partial y \partial x}(x,y)$ and $\frac{\partial^3 f}{\partial x \partial y \partial x}(x,y) = 12xy = \frac{\partial^3 f}{\partial y \partial x^2}(x,y)$, demonstrating the symmetry in mixed partial derivatives.

3.8 Functions of class C^p

Definition 3.11 (Functions of class C^p)**.** Let E be an open subset of $\mathbb{R}^n$ and p a positive integer. A function $f\colon E \to \mathbb{R}$ is said to be *of class* $C^p(E)$ if all its partial derivatives of order p exist and are continuous at every point in E.
A function $f\colon E \to \mathbb{R}$ is said to be of class $C^\infty(E)$ if, for every integer $p > 0$, it is of class $C^p(E)$.

Proposition 3.2. *If $f\colon E \to \mathbb{R}$ is a function of class $C^p(E)$, then it is also of class $C^k(E)$ for all $0 < k \leqslant p$.*

Example 3.8. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x\sin(xy)$. Then, for every $(x,y) \in \mathbb{R}^2$, we have:

$$\begin{aligned}
&\frac{\partial f}{\partial x}(x,y) = \sin(xy) + xy\cos(xy),\\
&\frac{\partial f}{\partial y}(x,y) = x^2\cos(xy),\\
&\frac{\partial^2 f}{\partial x^2}(x,y) = 2y\cos(xy) - xy^2\sin(xy),\\
&\frac{\partial^2 f}{\partial x\partial y}(x,y) = \frac{\partial^2 f}{\partial y\partial x}(x,y) = 2x\cos(xy) - x^2y\sin(xy),\\
&\frac{\partial^2 f}{\partial y^2}(x,y) = -x^3\sin(xy).
\end{aligned}$$

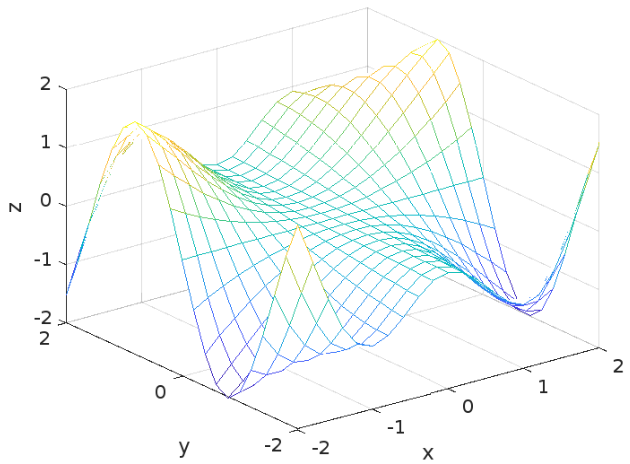


Figure 3.5: $f(x,y) = x\sin(xy)$

The following is a corollary of Schwarz's theorem.

Corollary 3.1. *Let $f\colon E \to \mathbb{R}$ be a function of class $C^p(E)$ and let k be an integer between 1 and p. If two ordered k-tuples $(i_1, \cdots, i_k)$ and $(j_1, \cdots, j_k)$ are equal up to* *a permutation, then, for any element* $\mathbf{a} = (a_1, \ldots, a_n)$ *of* E*, we have*

$$\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(a_1, \cdots, a_n) = \frac{\partial^k f}{\partial x_{j_1} \cdots \partial x_{j_k}}(a_1, \cdots, a_n).$$

3.9 Taylor's Theorem for Multivariable Functions

The following is a special (but often very useful) case of Taylor's theorem for multivariate functions.

Theorem 3.4 (Taylor's Formula – special case)**.** *Let* $E \subseteq \mathbb{R}^n$ *be open and* $f\colon E \to \mathbb{R}$ *a function of class* $C^{p+1}(E)$*. Then for every* $\mathbf{a} \in E$ *there exists a real number* $\delta > 0$ *such that* $B(\mathbf{a}, 2\delta) \subseteq E$ *and, for every element* $\mathbf{x} \in B(\mathbf{a}, \delta)$*, one can associate a number* $0 < \theta < 1$ *so that the following equality (known as Taylor's formula) holds:*

$$f(\mathbf{x}) = F(0) + F'(0) + \ldots + F^{(p)}(0)\frac{1}{p!} + F^{(p+1)}(\theta)\frac{1}{(p+1)!},$$

where $F\colon (-2, 2) \to \mathbb{R}$ *is the function defined by* $F(t) = f(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))$*.*

To state Taylor's theorem for multivariate functions in full generality, we first need to introduce the multi-index notation. Given an n-tuple of non-negative integers $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and a point $\mathbf{x} \in \mathbb{R}^n$, let

$$|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_n, \quad \boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_n!, \quad \mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

(Recall that by convention $0! = 1$.) For example, if $n = 3$ and $\boldsymbol{\alpha} = (1, 0, 4)$ then we have $|\boldsymbol{\alpha}| = 1 + 0 + 4 = 5$, and $\boldsymbol{\alpha}! = 1! \cdot 0! \cdot 4! = 24$, and $(x_1, x_2, x_3)^{\alpha} = x_1 x_3^4$. Given a function $f\colon E \to \mathbb{R}$ of class $C^k(E)$ and an n-tuple of non-negative integers $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with $|\boldsymbol{\alpha}| \leqslant k$ then we write

$$D^{\boldsymbol{\alpha}} f = \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Since f is of class $C^k(E)$, all its k-th order partial derivatives exist and are continuous and, by Schwarz's theorem, one can change the order of mixed derivatives. This ensures that as long as $|\boldsymbol{\alpha}| \leqslant k$ the above notation is well-defined and unambiguous.

Theorem 3.5 (Multivariate version of Taylor's theorem)**.** *Let* $k \in \mathbb{N}$*. Suppose* $E \subseteq \mathbb{R}^n$ *is open and* $f\colon E \to \mathbb{R}$ *is a function of class* $C^k(E)$*. Then*

$$f(\mathbf{x}) = \underbrace{\overbrace{\sum_{|\boldsymbol{\alpha}| \leqslant k} \frac{D^{\boldsymbol{\alpha}} f(\mathbf{a})}{\boldsymbol{\alpha}!}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}}^{k^{\text{th}}\text{-order approximation}} + \overbrace{r_k(\mathbf{x})}^{\text{remainder}}}_{k^{\text{th}}\text{-order expansion}} \tag{3.4}$$

where the sum is taken over all n*-tuples of non-negative integers* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *with* $|\boldsymbol{\alpha}| \leqslant k$ *and* $r_k(\mathbf{x})$ *is an "error" term satisfying* $\lim_{\mathbf{x} \to \mathbf{a}} \frac{r_k(\mathbf{x})}{\|\mathbf{x} - \mathbf{a}\|_2^k} = 0$.

Note that if $k = 1$ then formula (3.4) is the same as (3.2) and if $k = 2$ then formula (3.4) is the same as (3.3).

3.10 Local Extreme Values

One of the main uses of ordinary derivatives is in finding maximum and minimum values (extreme values). In this section we see how to use partial derivatives to locate maxima and minima of functions in more than one variables. This theory finds many applications, for example it can be used to maximize the volume of a box without a lid if we have a fixed amount of cardboard to work with.

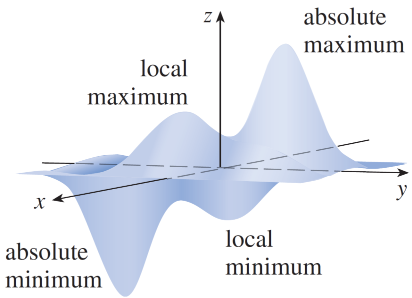


Figure 3.6

Look at the hills and valleys in the graph of f shown in Fig. 3.6. There are two points where f has a *local maximum*, that is, where f is larger than at nearby values, and two *local minima*, where f is smaller than at nearby values. We observe that at all these extreme values, the tangent plane to the graph is horizontal, or in other words, all the partial derivatives vanish at these points. This motivates the following definition.

Definition 3.12 (Stationary Point)**.** We say that $\mathbf{a} = (a_1, \ldots, a_n) \in E$ is a *stationary point* of the function $f\colon E \to \mathbb{R}$ if all its partial derivatives are well-defined and vanish at $\mathbf{a}$, that is,

$$\frac{\partial f}{\partial x_1}(a_1, \ldots, a_n) = \ldots = \frac{\partial f}{\partial x_n}(a_1, \ldots, a_n) = 0.$$

Definition 3.13 (Local Maximum and Minimum of a Function)**.** We say that the function $f\colon E \to \mathbb{R}$ admits a *local maximum* (resp. *local minimum*) at the point $\mathbf{a} \in E$ if there exists a real number $\delta > 0$ such that for all $\mathbf{x} \in E$ we have $\mathbf{x} \in B(a, \delta)$ implies $f(x) \leqslant f(a)$ (resp. $f(x) \geqslant f(a)$). Furthermore, we will say that a function admits a *local extreme value* at the point $\mathbf{a}$ if this function admits either a local maximum or a local minimum at that point.

The notion of a local maximum or minimum is not to be confused with the notion of (global) maximum or minimum given in Definition 2.10.

Theorem 3.6 (Necessary Condition for local extreme values)**.** *Let $f\colon E \to \mathbb{R}$ be a function and assume all partial derivatives of f at point $\mathbf{a}$ exist. If f has a local extreme value at the point $\mathbf{a}$, then $\mathbf{a}$ must be a stationary point.*

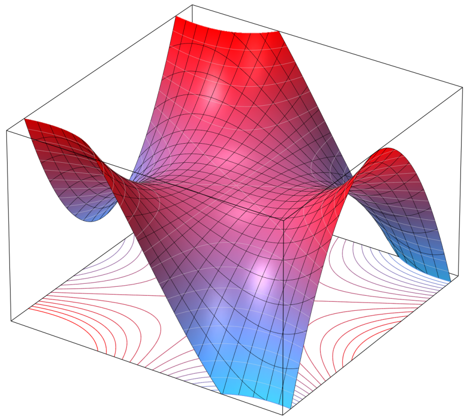


Figure 3.7: A so-called *monkey saddle surface*, with the equation $z = x^3 - 3xy^2$. Its name derives from the observation that a saddle for a monkey would require two depressions for the legs and one additional depression for the tail.

The geometric interpretation of Theorem 3.6 is that if the graph of f has a tangent plane and a local extreme value at a point $\mathbf{a}$, then this tangent plane must be horizontal.

Remark 3.5. The condition demonstrated in Theorem 3.6 is only a necessary one, but not sufficient, because stationary points are not always local extreme values. For example, let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = x^3 - 3xy^2$. Since

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0,$$

ti follows that $(0, 0)$ is a stationary point of f. However, f does not have a local extreme value at $(0, 0)$, which is evident from the graph of f depicted in Fig. 3.7. Indeed, we see that this surface has a horizontal tangent plane at the origin, yet it does not have a local extreme value at that point.

Proposition 3.3. *Given a function $f\colon E \to \mathbb{R}$, if f possesses a local extrema at the point $\mathbf{a} = (a_1, \ldots, a_n)$, then, in light of the necessary conditions outlined in Theorem 3.6, the point $\mathbf{a}$ must fall into one of the following categories:*

- *Stationary points of f, where the gradient of f exists and vanishes;*
- *Points within the domain E at which at least one of the partial derivatives of f does not exist.*

This categorization is crucial for identifying the points at which the function f may achieve its maximum or minimum values, highlighted by either a zero gradient (indicating a lack of change in all directions) or the absence of a derivative (indicative of a potential sharp point or discontinuity).

Example 3.9. Consider four points in $\mathbb{R}^2$: $A = (7,1)$, $B = (x,-x)$, $C = (y,y)$, and $D = (8,4)$. How should we choose x and y so that the sum of the distances from A to B, from B to C, and from C to D is minimal? This problem is equivalent to finding a point in $\mathbb{R}^2$ for which the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{aligned} f(x,y) &= \sqrt{(x-7)^2+(-x-1)^2}+\sqrt{(x-y)^2+(-x-y)^2}+\sqrt{(y-8)^2+(y-4)^2} \\ &= \sqrt{2}\left(\sqrt{x^2-6x+25}+\sqrt{x^2+y^2}+\sqrt{y^2-12y+40}\right) \end{aligned}$$

reaches its minimum. First, we need to demonstrate that such a point exists. For this, let $E = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leqslant 10^3\}$. Since f is continuous on E and E is a compact subset of $\mathbb{R}^2$, it follows from the Extreme Value Theorem (see Proposition 2.3) that there exists an element (a,b) in E such that

$$f(a,b) = \min_{(x,y)\in E} f(x,y).$$

Consequently, noting that for every $(x,y) \notin E$:

$$f(x,y) \geqslant \sqrt{2}\sqrt{x^2+y^2} > \sqrt{2}\sqrt{1000} > \sqrt{2}(5+\sqrt{40}) = f(0,0) \geqslant f(a,b)$$

we can conclude that

$$f(a,b) = \min_{(x,y)\in\mathbb{R}^2} f(x,y).$$

So there exists a global minimum for the function f. Notice that

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \sqrt{2}\left(\left(x^2-6x+25\right)^{-1/2}(x-3)+\left(x^2+y^2\right)^{-1/2}x\right) \\ \frac{\partial f}{\partial y}(x,y) &= \sqrt{2}\left(\left(x^2+y^2\right)^{-1/2}y+\left(y^2-12y+40\right)^{-1/2}(y-6)\right) \end{aligned}$$

for $(x,y) \neq (0,0)$. Since the only stationary point of f is $(1,2)$, we can assert that $(a,b) = (1,2)$ or $(a,b) = (0,0)$ (see Proposition 3.3). However,

$$f(1,2) = 5\sqrt{10} < \sqrt{2}(5+\sqrt{40}) = f(0,0)$$

thus, we can affirm that $(a,b) = (1,2)$. Consequently, the two sought points are

$$B = (1,-1) \text{ and } C = (2,2).$$

Fig. 3.8 below provides the geometric solution to this problem.

3.11 Global Extreme Values

The Extreme Value Theorem (Proposition 2.3) says that any continuous function on a compact set attains a maximum and minimum. To find these extreme values (which are sometimes also called absolute extreme values or global extreme values), we can employ the following extension of the Closed Interval Method.

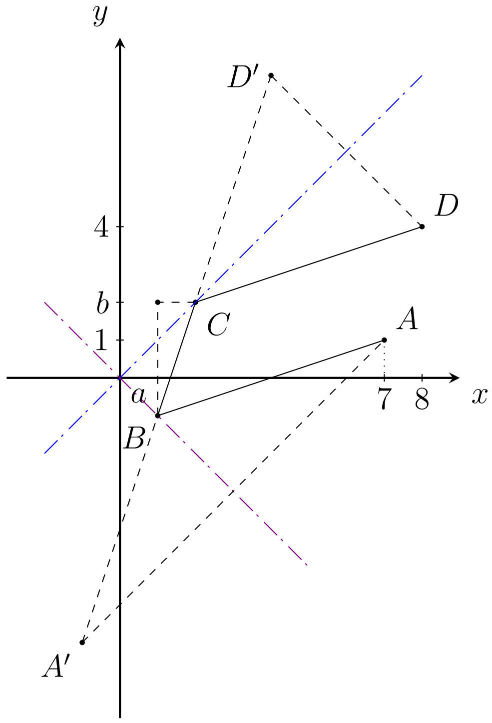


Figure 3.8: The aim is to find a point C on the blue line and B on the purple line such that the distance $\overline{AB} + \overline{BC} + \overline{CD}$ is minimal.

Finding Global Extreme Values. Let $f \colon E \to \mathbb{R}$ be a continuous function on a compact set E and suppose f is differentiable on the interior $\mathring{E}$. To find the absolute maximum and minimum values of f on E, complete the following three steps:

1. Find the stationary points of f on the interior $\mathring{E}$.
2. Find the extreme values of f on the boundary ∂E.
3. Compile a list of the function values at the points found in steps 1 and 2. The largest of these values is the (absolute/global) maximum value; the smallest of these values is the (absolute/global) minimum value.

Example 3.10. Let us find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) : 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 2\} = [0, 3] \times [0, 2]$.

Since f is a polynomial, it is continuous on the compact rectangle D, so Proposition 2.3 tells us there is both an absolute maximum and an absolute minimum. First we find all the stationary points. These occur when

$$\frac{\partial f}{\partial x}(x, y) = 2x - 2y = 0,$$
$$\frac{\partial f}{\partial y}(x, y) = 2x - 2y = 0,$$

so the only stationary point in is $(1, 1)$. This point is in $\mathring{D}$ and the value of f at this point is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D, which consists of the four line segments $L_1 = [0, 3] \times \{0\}$, $L_2 = \{3\} \times [0, 2]$, $L_3 = [0, 3] \times \{2\}$, and $L_4 = \{0\} \times [0, 2]$. On L_1 we have $y = 0$ and

$$f(x, 0) = x^2, \qquad 0 \leqslant x \leqslant 3.$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y, \qquad 0 \leqslant y \leqslant 2,$$

which is a decreasing function of y, so its minimum value is $f(3, 2) = 1$ and its maximum value is $f(3, 0) = 9$. On L_3 and L_4 we can execute very similar strategies. We find that when restricted to L_3, f has a minimum at $(2, 2)$, which is $f(2, 2) = 0$ and a maximum value at $(0, 2)$, which is $f(0, 2) = 4$. The maximum of f on L_4 is at $(0, 2)$, with $f(0, 2) = 4$, and the minimum is at $(0, 0)$ with $f(0, 0) = 0$.

In step 3, we compare all the values that we have thus far found:

(x,y)	f(x,y)
(1,1)	1
(0,0)	0
(3,0)	9
(3,2)	1
(2,2)	0
(0,2,)	4

We see that the maximum value of f on D is $f(3, 0) = 9$ and the minimum value is $f(0, 0) = f(2, 2) = 0$.

3.12 Saddle Points

Recall that for functions of a single variable, a stationary point c where $f'(c) = 0$ may correspond to a local maximum, a local minimum, or neither. An analogous situation occurs for multivariate functions. If $\mathbf{a}$ is a stationary point of a function f, where $\nabla f(\mathbf{a}) = \mathbf{0}$, then $f(\mathbf{a})$ may be a local maximum, a local minimum, or neither. In the last case, we are dealing with a so-called saddle point of f.

Definition 3.14. If $\mathbf{a}$ is a stationary point of a function f that is not a local extreme value then $\mathbf{a}$ is called a *saddle point* of f.

The name ‘saddle point’ derives from the fact that the prototypical example in two dimensions is a surface that curves up in one direction, and curves down in a different direction, resembling a riding saddle (for a rider of an animal such as a horse) or landform saddle (a mountain pass between two peaks). In general, the graph of a function at a saddle point need not resemble an actual saddle, but the graph crosses the tangent plane at that point.

In summary, saddle points are points where the tangent plane is horizontal, but there are points arbitrarily close to it where the function value lies above the tangent plane, and at the same time points arbitrarily close where the function value is below the tangent plane.

3.13 The Second Derivative Test – two-variable case

We need to be able to determine whether or not a function has an extreme value at a critical point. The following test is analogous to the Second Derivative Test for functions of one variable.

Theorem 3.7 (Second Derivative Test – 2 variable case)**.** *Let $E \subseteq \mathbb{R}^2$ be an open set and $f\colon E \to \mathbb{R}$ a function of class $C^2(E)$. Let D denote the determinant of the Hessian matrix of f at the point $(a,b) \in E$, i.e.,*

$$\begin{aligned} D = \det(\operatorname{Hess}(f)(a,b)) &= \left| \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial y \partial x}(a,b) \\ \frac{\partial^2 f}{\partial x \partial y}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{pmatrix} \right| \\ &= \frac{\partial^2 f}{\partial x^2}(a,b) \cdot \frac{\partial^2 f}{\partial y^2}(a,b) - \left(\frac{\partial^2 f}{\partial x \partial y}(a,b) \right)^2 . \end{aligned}$$

If (a,b) is a stationary point then the following conditions determine the nature of the extreme value at (a,b):

- *If $D > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) > 0$, then f has a local minimum at (a,b).*
- *If $D > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) < 0$, then f has a local maximum at (a,b).*
- *If $D < 0$, then f has a saddle point at (a,b).*
- *If $D = 0$ then the test is inconclusive.*

Remark 3.6.

- If $D = 0$ then the test gives no information: f could have a local maximum or local minimum or a saddle point at (a,b). An example of such a function would be $f(x,y) = (y-x^2)(y-2x^2)$ at the point $(a,b) = (0,0)$.
- If $D > 0$ then $\frac{\partial^2 f}{\partial x^2}(a,b)$ and $\frac{\partial^2 f}{\partial y^2}(a,b)$ are both non-zero and have the same sign. This means we can replace the condition $\frac{\partial^2 f}{\partial x^2}(a,b) > 0$ in the first part of the test with either the condition $\frac{\partial^2 f}{\partial y^2}(a,b) > 0$ or even with the condition $\operatorname{tr}(\operatorname{Hess}(f)(a,b)) > 0$, the trace of the Hessian matrix. The same goes with the condition $\frac{\partial^2 f}{\partial x^2}(a,b) > 0$ in the second part of the test.
- Note that Theorem 3.7 only concerns functions in two variables. There is also a version of the second derivative test for functions in three and more variables,

which we cover in the next section.

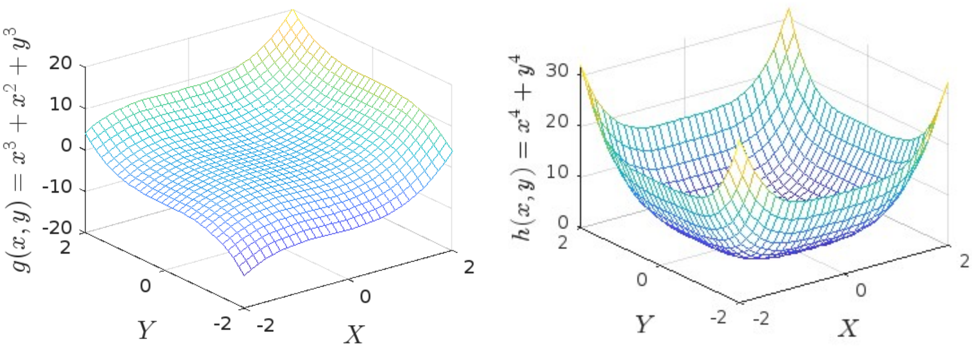
Example 3.11. The two functions $g, h : \mathbb{R}^2 \to \mathbb{R}$ defined respectively by $g(x, y) = x^3 + x^2 + y^3$ and $h(x, y) = x^4 + y^4$ (see Example 3.11) have $(0, 0)$ as a stationary point and satisfy

$$\left(\frac{\partial^2 g}{\partial x \partial y}(0,0)\right)^2 - \frac{\partial^2 g}{\partial x^2}(0,0) \cdot \frac{\partial^2 g}{\partial y^2}(0,0) = 0,$$
$$\left(\frac{\partial^2 h}{\partial x \partial y}(0,0)\right)^2 - \frac{\partial^2 h}{\partial x^2}(0,0) \cdot \frac{\partial^2 h}{\partial y^2}(0,0) = 0.$$

Since the function g does not have a local extreme value at the point $(0, 0)$, while the function h does, this example illustrates that for a C^2 class function $f : \mathbb{R}^2 \to \mathbb{R}$ in the neighborhood of (a, b) which satisfies

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0, \quad \text{and} \quad \left(\frac{\partial^2 f}{\partial x \partial y}(a,b)\right)^2 - \frac{\partial^2 f}{\partial x^2}(a,b) \cdot \frac{\partial^2 f}{\partial y^2}(a,b) = 0,$$

it is generally not possible a priori to determine whether it admits an extrema at the point (a, b).



Example 3.12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = y^3 + 3y^2 - 4xy + x^2$. Since for all $(x, y) \in \mathbb{R}^2$:

$$\frac{\partial f}{\partial x}(x,y) = -4y + 2x, \quad \frac{\partial f}{\partial y}(x,y) = 3y^2 + 6y - 4x,$$

and

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2, \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = -4, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 6(y+1),$$

it follows that the stationary points of the function f are $(0, 0)$ and $(4/3, 2/3)$, and at these points

$$\frac{\partial^2 f}{\partial x^2}(0,0) \cdot \frac{\partial^2 f}{\partial y^2}(0,0) - \left(\frac{\partial^2 f}{\partial x \partial y}(0,0)\right)^2 = -4 < 0,$$

and

$$\frac{\partial^2 f}{\partial x^2}(4/3, 2/3) \cdot \frac{\partial^2 f}{\partial y^2}(4/3, 2/3) - \left(\frac{\partial^2 f}{\partial x \partial y}(4/3, 2/3)\right)^2 = 4 > 0.$$

Therefore, according to Theorem 3.7, the function f has a local minimum at the point (4/3, 2/3), while at the point $(0, 0)$, it does not have a local extreme value because it is a saddle point.

3.14 The Second Derivative Test – general case

Recall from linear algebra that every real symmetric $n \times n$ matrix is diagonalizable. In particular, symmetric matrices possess n real eigenvalues (when counted with multiplicities) and admit a basis of eigenvectors. This also applies to the Hessian matrix of a function: As we have learned, if $f(x_1, \ldots, x_n)$ is a function in n variables of class C^2 then its Hessian matrix

$$\operatorname{Hess}(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

is a real symmetric matrix, which means it admits n real eigenvalues $\lambda_1, \ldots, \lambda_n$. These eigenvalues determine the curvature behavior of the function f and play a crucial role in the second derivative test for multivariate functions.

Theorem 3.8 (Second Derivative Test – general case)**.** *Let $E \subseteq \mathbb{R}^n$ be an open set and $f\colon E \to \mathbb{R}$ a function of class $C^2(E)$. Let $\mathbf{a} \in E$ and let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of the matrix* $\operatorname{Hess}(f)(\mathbf{a})$*. If $\mathbf{a}$ is a stationary point then the following conditions determine the nature of the extreme value at $\mathbf{a}$:*

- *If the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all positive then f has a local minimum at $\mathbf{a}$.*
- *If the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all negative then f has a local maximum at $\mathbf{a}$.*
- *If the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all non-zero, but some are positive and some are negative, then f has a saddle point at $\mathbf{a}$.*
- *If at least one of the eigenvalues $\lambda_1, \ldots, \lambda_n$ equals zero then the test is inconclusive.*

Example 3.13. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a function of class $C^2(\mathbb{R})$ and let $\mathbf{a}$ be a stationary point of f. If the three eigenvalues of the Hessian matrix $\operatorname{Hess}(f)(\mathbf{a})$ satisfy

$$\lambda_1 + \lambda_2 + \lambda_3 = 2 \qquad \text{and} \qquad \lambda_1 \lambda_2 \lambda_3 = -1$$

then can f have a local extreme value at the point $\mathbf{a}$? The answer is no. Since $\lambda_1 \lambda_2 \lambda_3 = -1$, the Second Derivative Test is not inconclusive, so we must be either in the first, second, or third case of the test. However, since $\lambda_1 \lambda_2 \lambda_3$ is negative we cannot be in the first case, and since $\lambda_1 + \lambda_2 + \lambda_3$ is positive we cannot be in the second case. By method of elimination, we must be in the third case of the test, so $\mathbf{a}$ is a saddle point of f.

3.15 Implicit Function Theorem

In mathematics, we say that variables are in an *explicit relation* when one variable is expressed directly in terms of the other variable(s). For example, an *explicit equation* of a variable x_n in terms of the variables $x_1, \ldots, x_{n-1}$ is a relation of the form

$$x_n = f(x_1, \ldots, x_{n-1}),$$

where f is a function of $n-1$ variables. In this context, we refer to x_n as the *dependent variable* and $x_1, \ldots, x_{n-1}$ as the *independent variables* and the function f is the "law" that describes the relationship between x_n and $x_1, \ldots, x_{n-1}$. The great advantage of explicit relations is that if one knows the values of all the independent variables $x_1, \ldots, x_{n-1}$ then it is relatively easy to calculate the values of the dependent variable x_n.

In contrast to explicit relations, variables can also be in an *implicit relation*, which means their relationship isn't expressed explicitly in terms of one variable depending on the others. More precisely, an *implicit equation* in the variables $x_1, \ldots, x_n$ is a relation of the form

$$F(x_1, \ldots, x_n) = c,$$

where F is a function of n variables and $c \in \mathbb{R}$ is a constant. For example, the unit circle is commonly described by the implicit equation

$$x^2 + y^2 = 1.$$

Note that simple implicit equations can easily be transformed into explicit equations by isolating one variable on one side of the equation. For example, the implicit equation $x + y + z = 1$ (which describes a plane in $\mathbb{R}^3$) can easily be tuned into the explicit equation $z = 1 - x - y$ using rudimentary algebraic manipulations. But if the implicit equation is more complicated then it is often not possible to express one variable in terms of the others by hand. In this case, we need a more sophisticated tool, which is where the Implicit Function Theorem comes into play.

An *implicit function* is a function defined by an implicit equation that expresses one of the variables, say x_n, as a function of other variables, say $x_1, \ldots, x_{n-1}$. Here's the simple example: The equation $x^2 + y^2 = 1$ of the unit circle defines y as an implicit function of x if $-1 < x < 1$, and y is restricted to positive values. Under this restrictions we have

$$\underbrace{x^2 + y^2 = 1}_{\textit{implicit equation}} \qquad \Longleftrightarrow \qquad \underbrace{y = \sqrt{1 - x^2}}_{\textit{implicit function for } y>0},$$

where $f(x) = \sqrt{1 - x^2}$ is the implicit function defined by the implicit equation $x^2 + y^2 = 1$ in the domain $\{(x, y) : -1 < x < 1,\ y > 0\}$. Similarly, if y is restricted to negative values then we have

$$\underbrace{x^2 + y^2 = 1}_{\text{implicit equation}} \qquad \Longleftrightarrow \qquad \underbrace{y = -\sqrt{1-x^2}}_{\text{implicit function for } y<0} \quad ,$$

where $f(x) = -\sqrt{1-x^2}$ is the implicit function defined by the implicit equation $x^2 + y^2 = 1$ in the domain $\{(x, y) : -1 < x < 1,\ y < 0\}$. If $y = 0$, or equivalently if $x = 1$ or $x = -1$, then it is impossible to express y in terms of x and so the implicit function does not exist.

The Implicit Function Theorem tells under what conditions – and in what neighborhood – an implicit function exists, which helps us deal with cases where we have an implicit equation relating multiple variables and it's not easy to solve explicitly for one variable in terms of the others.

Theorem 3.9 (Implicit Function Theorem)**.** *Let n be an integer where $n \geqslant 2$. Let $E \subseteq \mathbb{R}^n$ be an open set, and let $F\colon E \to \mathbb{R}$ be a function of class $C^1(E)$. If $\mathbf{a} = (a_1, \dots, a_n) \in E$ and $c \in \mathbb{R}$ is such that*

$$F(\mathbf{a}) = c \quad \text{and} \quad \frac{\partial F}{\partial x_n}(\mathbf{a}) \neq 0,$$

then there exist a neighborhood $U \subseteq \mathbb{R}^{n-1}$ of the point $(a_1, \dots, a_{n-1})$, a neighborhood $V \subseteq \mathbb{R}$ of the point a_n, and a unique function $f\colon U \to V$ such that for all $(x_1, \dots, x_{n-1}) \in U$ and all $x_n \in V$ we have

$$F(x_1, \dots, x_n) = c \qquad \Longleftrightarrow \qquad x_n = f(x_1, \dots, x_n).$$

The function $f\colon U \to V$ is called the implicit function for the equation $F(x_1, \dots, x_n) = c$ at the point $(a_1, \dots, a_n)$.

Remark 3.7. Note that the implicit function $f\colon U \to V$ satisfies

$$a_n = f(a_1, \dots, a_{n-1}).$$

This follows from the assumption $F(a_1, \dots, a_n) = c$.

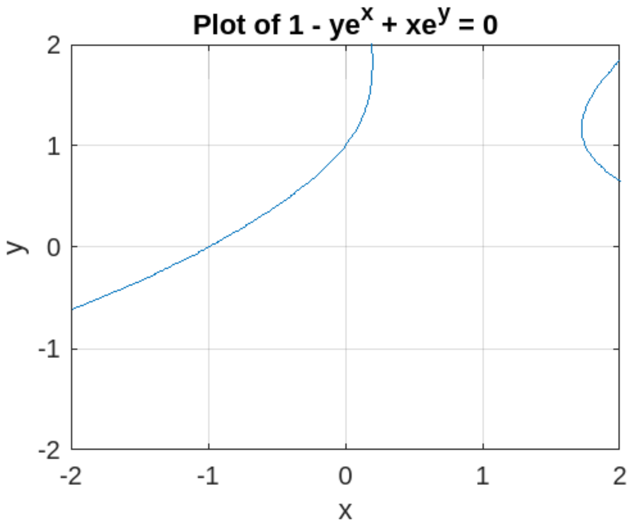
Remark 3.8. If, in the statement of the Implicit Function Theorem Theorem 3.9, we do not assume that $\frac{\partial F}{\partial x_n}(\mathbf{a}) \neq 0$, then the result may no longer be true, even if the other assumptions are satisfied. For example, this is the case for the function $F\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y) = x^2 + y^2$ for $\mathbf{a} = (0, 0)$.

Example 3.14. Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $F(x, y) = 1 - ye^x + xe^y$. Since $F(0, 1) = 0$ and $\frac{\partial F}{\partial y}(0, 1) = -1$, we know, thanks to the Implicit Function Theorem, that there exists a real number $\delta > 0$ and a continuously differentiable function $f\colon (-\delta, \delta) \to \mathbb{R}$ satisfying the following two properties (see Example 3.14): $f(0) = 1$ and $F(x, f(x)) = 0$ for every $x \in (-\delta, \delta)$. Since the derivative of the function $s \to F(s, f(s))$ is zero, we can use the chain rule for multivariable functions (which we will cover in Section 5.6) to conclude that

$$\frac{\partial F}{\partial x}(0,1) + \frac{\partial F}{\partial y}(0,1)f'(0) = 0$$

and therefore

$$f'(0) = -\frac{\frac{\partial F}{\partial x}(0,1)}{\frac{\partial F}{\partial y}(0,1)} = -1 + e.$$



3.16 Implicit Differentiation

The technique we used at the end of Example 3.14 to compute the derivative of a function is called implicit differentiation.

Theorem 3.10 (Implicit differentiation)**.** *Let n be an integer, where $n \geqslant 2$, let E be an open subset of $\mathbb{R}^n$, and let $F\colon E \to \mathbb{R}$ be a function of class $C^1(E)$. Suppose $\mathbf{a} = (a_1, \ldots, a_n)$ and there exists exists a real number $\delta > 0$ and a function $f\colon B((a_1, \ldots, a_{n-1}), \delta) \to \mathbb{R}$ of class $C^1(B((a_1, \ldots, a_{n-1}), \delta))$ such that*

$$F(x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1})) = 0$$

holds for all $(x_1, \ldots, x_{n-1}) \in B((a_1, \ldots, a_{n-1}), \delta)$. Then

$$\frac{\partial f}{\partial x_j}(a_1, \ldots, a_{n-1}) = -\frac{\frac{\partial F}{\partial x_j}(\mathbf{a})}{\frac{\partial F}{\partial x_n}(\mathbf{a})}, \qquad \forall j = 1, \ldots, n-1.$$

3.17 Tangent Line to Implicit Curves

An *implicit curve* is a plane curve defined by an implicit equation relating two variables, commonly x and y. For example, the unit circle is defined by the implicit equation

$x^2 + y^2 = 1$. In general, every implicit curve is defined by an equation of the form

$$F(x, y) = c$$

for some function F of two variables and some constant c. Hence an implicit curve can always be considered as the level curve of a function in two variables (cf. Definition 2.2). In this context, "implicit" means that the equation is not expressed explicitly in either one of the variables of the function.

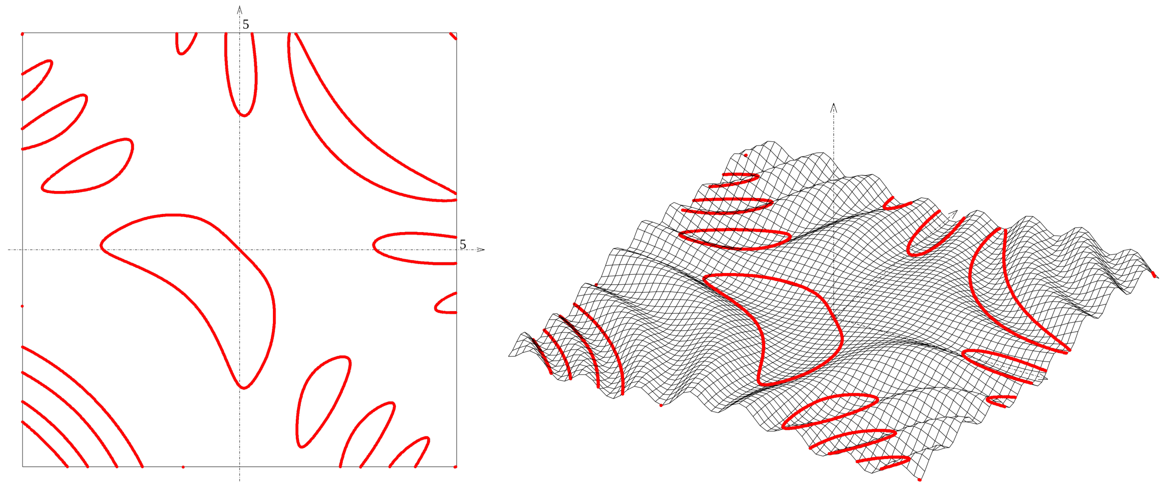


Figure 3.9: The implicit curve $\sin(x + y) - \cos(xy) + 1 = 0$ plotted as a graph in 2 dimensions (left) and as a level curve of the surface $z = \sin(x + y) - \cos(xy) + 1$ in 3 dimensions (right). This example also showcases the possibly complicated geometric structure of an implicit curve.

Let $D \subseteq \mathbb{R}^2$ be an open set, $F \colon D \to \mathbb{R}$ a function of class $C^1(D)$, $c \in \mathbb{R}$, and consider the implicit curve defined by the equation

$$F(x, y) = c, \quad (x, y) \in D.$$

The implicit function theorem (Theorem 3.9) describes conditions under which the above equation can be solved in terms of x and/or y. This theorem is key for the computation of essential geometric features of implicit curves such as tangents, normal vectors, and curvature. In particular, the Implicit Function Theorem says that if $(a, b) \in D$ such that

$$F(a, b) = c \quad \text{and} \quad \frac{\partial F}{\partial y}(a, b) \neq 0,$$

then there exists a function f such that for all points $(x, y) \in D$ with $\|(x, y) - (a, b)\|$ sufficiently small, we have $F(x, y) = c \iff y = f(x)$. This leads to two crucial insights:

- **Equivalence between the level set and the graph of f:** If the point (x, y) is sufficiently close to (a, b) then it satisfies the equation $F(x, y) = c$ if and only if it lies on the graph of the function f. Formally, this relationship is expressed as:

$$(x,y) \in L_c(F) \iff (x,y) \in G(f),$$

where $L_c(F) = \{(x,y) \in D : F(x,y) = c\}$ denotes the level set of F at height c, and $G(f) = \{(x, f(x)) : x \in \mathrm{dom}(f)\}$ represents the graph of the function f.

- **Tangent line equation at a point on the graph of f:** Recall from your Analysis I course that the tangent to the graph of f at the point (a,b) is given by the equation

$$y = f(a) + f'(a) \cdot (x - a).$$

By implicit differentiation (Theorem 3.10) we know that

$$f'(a) = -\frac{\frac{\partial F}{\partial x}(a,b)}{\frac{\partial F}{\partial y}(a,b)},$$

which allows us to rewrite the equation of the tangent line as

$$y = f(a) - \frac{\frac{\partial F}{\partial x}(a,b)}{\frac{\partial F}{\partial y}(a,b)} \cdot (x - a).$$

Finally, using $f(a) = b$, we can express the tangent line of F at the point (a,b) in terms of the gradient as

$$\nabla F(a,b) \cdot \begin{pmatrix} x - a \\ y - b \end{pmatrix} = 0.$$

Equation of the tangent line to an implicit curve. Let $D \subseteq \mathbb{R}^2$ be an open set, $F \colon D \to \mathbb{R}$ a function of class $C^1(D)$, and $c \in \mathbb{R}$ a real number. Consider the implicit curve defined by the equation $F(x,y) = c$. If (a,b) is a point on this curve with $\nabla F(a,b) \neq \mathbf{0}$ then the equation of the tangent line to this implicit curve at the point (a,b) is

$$\nabla F(a,b) \cdot \begin{pmatrix} x - a \\ y - b \end{pmatrix} = 0.$$

Example 3.15. Given $c > 0$, let us find the tangent line to the circle $x^2 + y^2 = c$ at a point (a,b) on this circle.

Letting $F(x,y) = x^2 + y^2$, the level set $L_c(F)$ is a circle of radius $\sqrt{c}$. For a point (a,b) such that $a^2 + b^2 = c$ and $b \neq 0$, the condition $\frac{\partial F}{\partial y}(a,b) = 2b \neq 0$ holds. Thus, near (a,b), the level set $L_c(F)$ corresponds to the graph of the function $x \mapsto f(x)$, defined as $f(x) = \sqrt{c - x^2}$. If $a^2 + b^2 = c$ with $b = 0$, we can swap the roles of x and y, as then

$$\frac{\partial F}{\partial x}(a,b) = 2a \neq 0.$$

In either one of the two cases, the gradient of F is $\nabla F(x, y) = (2x, 2y)$ and hence $\nabla F(a, b) = (2a, 2b)$. Therefore, the equation of the line through the point (a, b) and tangent to the circle $x^2 + y^2 = c$ is

$$(2a, 2b) \cdot \begin{pmatrix} x - a \\ y - b \end{pmatrix} = 0.$$

Using $a^2 + b^2 = c$, this can be simplified to

$$ax + by = c.$$

3.18 Tangent Plane to Implicit Surfaces

An *implicit surface* is a surface in $\mathbb{R}^3$ defined by an equation of the form

$$F(x, y, z) = d,$$

where F is some function depending on three variables and d is some constant real number. Implicit surfaces are the same as level surfaces of functions in three variables.

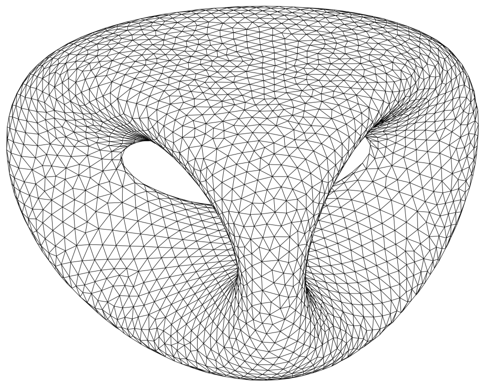


Figure 3.10: The surface that is depicted above is defined by the implicit equation $2y(y^2 - 3x^2)(1 - z^2) + (x^2 + y^2)^2 - (9z^2 - 1)(1 - z^2) = 0$.

Let $D \subseteq \mathbb{R}^3$ be an open set, $F\colon D \to \mathbb{R}$ be a function of class $C^1(D)$, and $(a, b, c) \in D$ with $d \in \mathbb{R}$ such that

$$F(a, b, c) = d \quad \text{and} \quad \frac{\partial F}{\partial z}(a, b, c) \neq 0.$$

The Implicit Function Theorem guarantees the existence of a differentiable function f such that for all

$$c = f(a, b) \quad \text{and} \quad F(x, y, f(x, y)) = d \quad \text{for all } (x, y) \text{ sufficiently close to } (a, b).$$

- **First consequence:** For any (x, y, z) sufficiently close to (a, b, c), we have :

$$F(x, y, z) = d \iff z = f(x, y).$$

In other words, locally around the point (a, b, c) the level set $L_d(F)$ and the graph $G(f)$ coincide.

- **Second consequence:** As we have learned in Section 3.4, the equation of the tangent plane to the graph of f at (a, b) is given by

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

On the other hand, using implicit differentiation (Theorem 3.10) we have

$$\frac{\partial f}{\partial x}(a,b) = -\frac{\frac{\partial F}{\partial x}(a,b,c)}{\frac{\partial F}{\partial z}(a,b,c)} \qquad \text{and} \qquad \frac{\partial f}{\partial y}(a,b) = -\frac{\frac{\partial F}{\partial y}(a,b,c)}{\frac{\partial F}{\partial z}(a,b,c)}.$$

So we can rewrite the equation of the tangent plane in terms of the gradient of F as

$$\nabla F(a,b,c) \cdot \begin{pmatrix} x-a \\ y-b \\ z-c \end{pmatrix} = 0,$$

which is the equation of the tangent plane to the graph of f at the point (a, b, c). Thus, $\nabla F(a, b, c)$ is orthogonal to the tangent plane of the graph of f at (a, b, c).

Equation of the tangent plane to an implicit surface. Let $D \subseteq \mathbb{R}^3$ be an open set, $F : D \to \mathbb{R}$ a function of class $C^1(D)$, and $d \in \mathbb{R}$ a real number. Consider the implicit curve defined by the equation $F(x, y, z) = d$. If (a, b, c) is a point on this curve with $\nabla F(a, b, c) \neq \mathbf{0}$ then the equation of the tangent plane to this implicit surface at the point (a, b, c) is

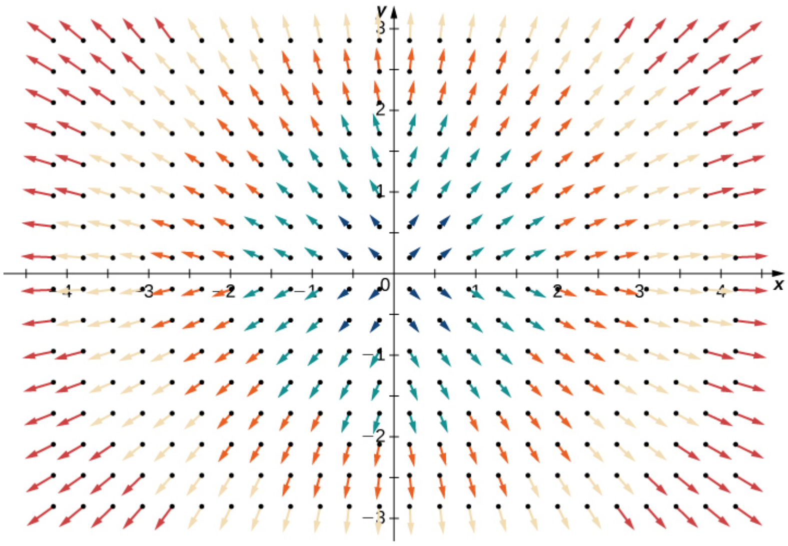
$$\nabla F(a,b,c) \cdot \begin{pmatrix} x-a \\ y-b \\ z-c \end{pmatrix} = 0.$$

Example 3.16. Let $F(x, y, z) = x^2 + y^2 + z^2$ and consider the level set $F(x, y, z) = 1$, which describes a sphere of radius 1. For a point (x_0, y_0) such that $x_0^2 + y_0^2 < 1$, let $z_0 = \pm\sqrt{1 - x_0^2 - y_0^2}$. We have $F(x_0, y_0, z_0) = 1$ and $\frac{\partial F}{\partial z}(x_0, y_0, z_0) = 2z_0 \neq 0$. The equation of the tangent plane at the point (x_0, y_0, z_0) is given by:

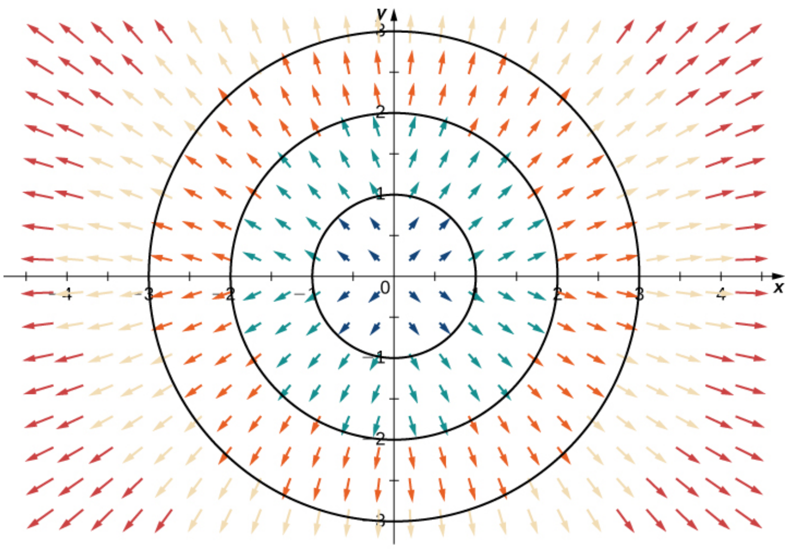
$$\nabla F(x_0,y_0,z_0) \cdot \begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix} = 0 \iff \begin{pmatrix} 2x_0 \\ 2y_0 \\ 2z_0 \end{pmatrix} \cdot \begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix} = 0.$$

Simplifying the expression and using $x_0^2 + y_0^2 + z_0^2 = 1$ we get the euqation of the tangent plane as

$$x_0 x + y_0 y + z_0 z = 1.$$



(a)



(b)

3.19 Method of Lagrange Multipliers – single constraint

Constrained optimization is the process of optimizing a function with respect to some variables in the presence of constraints on those variables. The Method of Lagrange Multipliers is a powerful technique for constrained optimization. It lets you find the maximum or minimum of a multivariable function subject to an implicit constraint equation. While it was originally developed to solve physics equations, today it finds applications in all sciences, especially in machine learning. To motivate the subject matter, let us first look at a simple constrained optimization problem that you are probably familiar with from your high school mathematics education.

Example 3.17. For a rectangle whose perimeter is 20 meters, find the dimensions that will maximize the area.

Solution: Let x denote the width and y the height of the rectangle in question. Both the area $A(x, y) = xy$ and the perimeter $P(x, y) = 2x + 2y$ of the rectangle are functions in the two variables x and y. The constrained optimization problem can now be summarized as:

$$\begin{aligned} \text{Maximize}:\ & A(x, y), \\ \text{Constraint}:\ & P(x, y) = 20. \end{aligned}$$

There is a simple method, using single-variable calculus, for solving this problem. Since the implicit equation $2x + 2y = 20$ can easily be recast as an explicit equation $y = 10 - x$, we can substitute this explicit formula into $A(x, y)$ to get a new function $f(x) = A(x, 10 - x) = 10x - x^2$. This is now a function of x alone, so we just have to maximize the function $f(x) = 10x - x^2$ on the interval $[0, 10]$. Since $f'(x) = 10 - 2x$ we see that $x = 5$ is a stationary point for $f(x)$. Since $f''(5) = -2 < 0$, the Second Derivative Test tells us that $x = 5$ is a local maximum for f, and hence $x = 5$ must be the global maximum on the interval $[0, 10]$ (since the interval is compact and the function f equals 0 at the endpoints of the interval). So since $y = 10 - x = 5$, then the maximum area occurs for a rectangle whose width and height are both equal to 5 meters.

Notice in the above example that the ease of the solution depended on being able to solve the constraint equation for one variable in terms of the other. However, this is not always possible, especially when the constraint equation is more complicated and when there are more variables involved. In this case, the hands-on task of solving the constraint equation in terms of one of the variables is replaced by an application of the Implicit Function Theorem.

The general type of constrained optimization problem that we are interested in is:

$$\begin{aligned} \text{Maximize (or minimize)}:\ & f(x_1, \ldots, x_n), \\ \text{Constraint}:\ & g(x_1, \ldots, x_n) = c. \end{aligned}$$

The function being maximized or minimized, $f(x_1, \ldots, x_n)$, is called the *objective function*. The function, $g(x_1, \ldots, x_n)$, whose level set at height c represents the constraint, that is, all the values allowed to be considered for the optimization, is called the *constraint function*. Points $(x_1, \ldots, x_n)$ which yield maxima or minima of $f(x_1, \ldots, x_n)$ with the condition that they satisfy the constraint equation $g(x_1, \ldots, x_n) = c$ are called *constrained maximum points* or *constrained minimum points*, respectively.

A constrained optimization problem in two variables has an illustrative geometric interpretation. Indeed, if the input space is two-dimensional, then the graph of the objective function $f(x, y)$ is a 3 dimensional surface and the constraint equation $g(x, y) = c$ is a curve in 2 dimensions. We can projected the curve (in red) onto the surface (in blue) as shown in Fig. 3.11. The goal of the constrained optimization problem is simply to find the highest (resp. lowest) point on that red line.

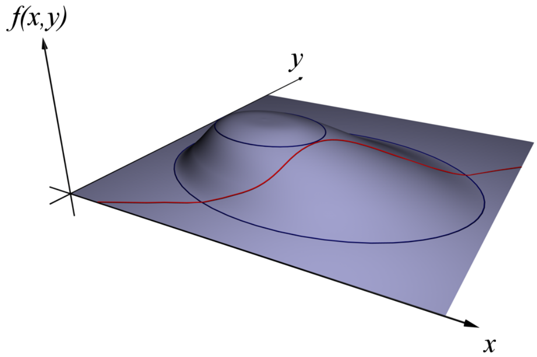


Figure 3.11: Constrained optimization problem in two variables.

In Fig. 3.11 we see that the highest point on the red line is the point where the red line is tangent to a level curve of $f(x, y)$. But the red line is itself a level curve coming from the function $g(x, y)$. So the core idea is to look for points where the level curves of $f(x, y)$ and $g(x, y)$ are tangent. This is the same as finding points where the gradient vectors ∇f and ∇g are are parallel to each other (see Fig. 3.12). In other words, there exists some $\lambda \in \mathbb{R}$ such that $\nabla f = \lambda \nabla g$.

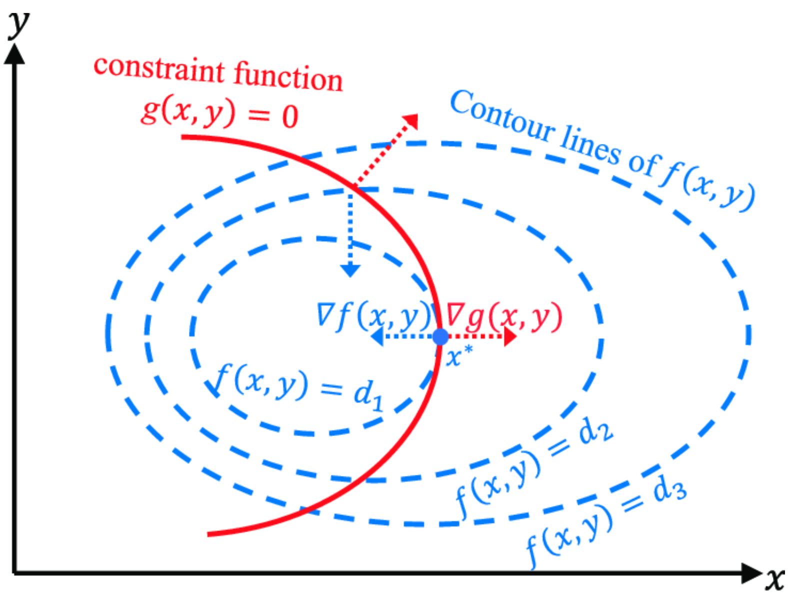


Figure 3.12: Maximization of function $f(x, y)$ subject to the constraint $g(x, y) = 0$. At the constrained local extreme value, the gradients of f and g, namely $\nabla f(x, y)$ and $\nabla g(x, y)$, are parallel.

In general, the Lagrange multiplier method for solving constrained optimization problems can be stated as follows.

Theorem 3.11 (Lagrange Multiplier Theorem)**.** *Consider an open set $E \subseteq \mathbb{R}^n$, two functions $f, g\colon E \to \mathbb{R}$ of class $C^1(E)$ and let $c \in \mathbb{R}$ be a constant. If the function f*

restricted to the level set $\{\mathbf{x} \in E : g(\mathbf{x}) = c\}$ *achieves a local extreme value at a point* $\mathbf{a}$ *and additionally* $\nabla g(\mathbf{a}) \neq \boldsymbol{0}$ *then there must be a scalar number* $\lambda \in \mathbb{R}$ *such that* $\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a})$. *The number* λ *is called the* Lagrange multiplier.

Example 3.18. For a rectangle whose perimeter is 20 m, use the Lagrange multiplier method to find the dimensions that will maximize the area.

Solution: As we saw in Example 3.17, with x and y representing the width and height, respectively, of the rectangle, this problem can be stated as:

$$\begin{aligned} \text{Maximize :} \quad & A(x,y) = xy \\ \text{Constraint equation :} \quad & P(x,y) = 2x + 2y = 20 \end{aligned}$$

In light of Theorem 3.11, the above can only have a solution when $\nabla A(x,y) = \lambda \nabla P(x,y)$ for some λ. Since $\nabla A(x,y) = (y,x)$ and $\nabla P(x,y) = (2,2)$, we need to solve the system of equations

$$\begin{aligned} y &= 2\lambda \,, \\ x &= 2\lambda. \end{aligned}$$

The general idea is to solve for λ in both equations, then set those expressions equal (since they both equal λ) to solve for x and y. Doing this we get

$$\frac{y}{2} = \lambda = \frac{x}{2} \quad \Longrightarrow \quad x = y.$$

Substituting either of the expressions for x or y into the constraint equation, we obtain

$$20 = g(x,y) = 2x + 2y = 2x + 2x = 4x \quad \Longrightarrow \quad x = 5 \quad \Longrightarrow \quad y = 5.$$

Hence there must be a maximum area, since the minimum area is 0 and $f(5,5) = 2S > 0$, so the point $(5,5)$ that we found (called a *constrained critical point*) must be the constrained maximum. Therefore the maximum area occurs for a rectangle whose width and height both are 5 meters.

Example 3.19. Let us find the constrained extreme values of the expression $x + z$ subject to the constrained $g(x,y,z) = x^2 + y^2 + z^2 = 1$. In other words,

$$\begin{aligned} \text{Maximize (and minimize) :} \quad & f(x,y,z) = x + z, \\ \text{Constrained equation :} \quad & g(x,y,z) = x^2 + y^2 + z^2 = 1. \end{aligned}$$

By Theorem 3.11, the strategy is to look for solutions to the equation $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$. Since $\nabla f(x,y,z) = (1,0,1)$ and $\nabla g(x,y,z) = (2x,2y,2z)$, we have

$$\begin{aligned} 1 &= 2\lambda x \\ 0 &= 2\lambda y \\ 1 &= 2\lambda z \end{aligned}$$

The first equation implies $\lambda \neq 0$ (otherwise we would have $1 = 0$), so we can divide by λ in the second equation to get $y = 0$ and we can divide by λ in the first and third equations to get $x = \frac{1}{2\lambda} = z$. Substituting these expressions into the constraint equation $g(x, y, z) = x^2 + y^2 + z^2 = 1$ yields the constrained critical points $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ and $(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$. Since $f(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) > f(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$, and since the constraint equation $x^2 + y^2 + z^2 = 1$ describes a sphere (which is bounded) in $\mathbb{R}^3$, then $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ is the constrained maximum point and $(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$ is the constrained minimum point.

Example 3.20. We aim to prove that for any m-tuple of positive real numbers $(\alpha_1, \ldots, \alpha_m)$, the following inequality holds:

$$\sqrt[m]{\alpha_1 \cdot \ldots \cdot \alpha_m} \leqslant \frac{\alpha_1 + \ldots + \alpha_m}{m}.$$

In other words, the geometric mean of a finite number of elements from $\mathbb{R}_+^*$ is never greater than their arithmetic mean.

Given an arbitrary m-tuple of positive real numbers

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m),$$

let us consider the set

$$E = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_1 \geqslant 0, \ldots, x_m \geqslant 0\}$$

and define two functions $f, g : E \to \mathbb{R}$ by

$$f(x_1, \ldots, x_m) = \sqrt[m]{x_1 \cdot \ldots \cdot x_m},$$

$$g(x_1, \ldots, x_m) = x_1 + \ldots + x_m - \beta, \quad \text{where } \beta = \alpha_1 + \ldots + \alpha_m.$$

Given that

$$E_1 = \{(x_1, \ldots, x_m) \in E \mid g\,(x_1, \ldots, x_m) = 0\}$$

is a compact subset of $\mathbb{R}^m$ and f is continuous, there exists at least one element $\mathbf{a} = (a_1, \ldots, a_m)$ in E_1 where the restriction of f to E_1 achieves its maximum. The method of Lagrange multipliers asserts that this maximum, referred to as a constrained maximum, occurs in the following cases:

1) $a_1 \cdot \ldots \cdot a_m = 0$,
2) $a_1 \cdot \ldots \cdot a_m > 0$ and there exists a real number λ such that

$$\begin{cases} \dfrac{\partial f}{\partial x_1}(\mathbf{a}) + \lambda \dfrac{\partial g}{\partial x_1}(\mathbf{a}) = 0, \\ \qquad\qquad\vdots \\ \dfrac{\partial f}{\partial x_m}(\mathbf{a}) + \lambda \dfrac{\partial g}{\partial x_m}(\mathbf{a}) = 0. \end{cases}$$

In the first case, we have $f(\mathbf{a}) = 0$. Observing that $(\frac{\beta}{m}, \ldots, \frac{\beta}{m}) \in E_1$ and that $f(\frac{\beta}{m}, \ldots, \frac{\beta}{m}) > 0$, we conclude that the first case does not occur for a point a where the constrained maximum is achieved. In the second case, f is indeed of class C^1 in the vicinity of a and there exists a real number λ such that

$$\begin{cases} \dfrac{1}{m}\dfrac{\sqrt[m]{a_1\cdot\ldots\cdot a_m}}{a_1}+\lambda=0, \\ \qquad\qquad\vdots \\ \dfrac{1}{m}\dfrac{\sqrt[m]{a_1\cdot\ldots\cdot a_m}}{a_m}+\lambda=0. \end{cases}$$

Therefore, by solving this system and taking into account that $a_1+\ldots+a_m=\beta$, we deduce that

$$a_1=\ldots=a_m=\frac{\beta}{m},$$

and thus the constrained maximum is achieved at $\mathbf{a}=\left(\frac{\beta}{m},\ldots,\frac{\beta}{m}\right)\in E_1$. Finally, since $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_m)\in E_1$, we can state that

$$\sqrt[m]{\alpha_1\cdot\ldots\cdot\alpha_m}=f(\boldsymbol{\alpha})\leqslant f(\mathbf{a})=\sqrt[m]{a_1\cdot\ldots\cdot a_m}=\frac{\beta}{m}=\frac{\alpha_1+\ldots+\alpha_m}{m}.$$

Example 3.21. Consider a situation in which $\nabla g(x_0,y_0)\neq 0$ is not satisfied, and thus the theorem cannot be applied to the functions $f(x,y)=x^2+y$ and $g(x,y)=y^2$. Clearly, f admits a local minimum at $(x_0,y_0)=(0,0)$ under the constraint $g(x_0,y_0)=0$, since $f(x,0)=x^2$.

Moreover, we have

$$\nabla f(x,y)=\begin{pmatrix}2x\\1\end{pmatrix},\quad \nabla g(x,y)=\begin{pmatrix}0\\2y\end{pmatrix},$$

$$\nabla f(0,0)=\begin{pmatrix}0\\1\end{pmatrix},\quad \nabla g(0,0)=\begin{pmatrix}0\\0\end{pmatrix},$$

hence there exists no $\lambda\in\mathbb{R}$ such that

$$\nabla f(0,0)=\lambda\nabla g(0,0).$$

Here, $\nabla g(x_0,y_0)\neq 0$ is not satisfied.

In fact, $\nabla g(x,y)=\begin{pmatrix}0\\2y\end{pmatrix}$ for all $(x,y)\in\mathbb{R}^2$ st. $g(x,y)=0$.

Intuitive Explanation for the Theorem: We argue by contradiction and assume that the calculation is false. That is, $\nabla f(x_0,y_0)$ is not a multiple of $\nabla g(x_0,y_0)$ (in particular $\nabla f(x_0,y_0)\neq 0$). Fix $c=f(x_0,y_0)\in\mathbb{R}$. Since $\nabla f(x_0,y_0)$ is orthogonal to the level set $L_c(f)$ at (x_0,y_0), and $\nabla g(x_0,y_0)$ is orthogonal to the level set $L_0(g)$ at (x_0,y_0), we deduce that $L_c(f)$ crosses $L_0(g)$ without being tangent to it. This implies that for $\varepsilon>0$ small enough, $L_0(g)$ also crosses $L_{c+\varepsilon}(f)$ and $L_{c-\varepsilon}(f)$. In particular, f does not have a local extremum at (x_0,y_0).

Example 3.22. Consider a box without a cover.

$$\begin{aligned}\text{Volume} &= abc,\\ \text{Surface Area} &= ab + 2ac + 2bc.\end{aligned}$$

Find all those boxes of maximal volume for a given surface area $S > 0$. We put $x = ab, y = ac, z = bc$, so $f(x, y, z) = \sqrt{xyz}$ represents the volume.

$$g(x, y, z) = x + 2y + 2z - S = 0, \quad E = \left\{(x, y, z) \in \mathbb{R}^3 : x \geqslant 0, y \geqslant 0, z \geqslant 0\right\}.$$

For x, y, z, we recover a, b, c as: $a = \sqrt{\frac{xy}{z}}, b = \sqrt{\frac{xz}{y}}, c = \sqrt{\frac{yz}{x}}$. Moreover, $xyz = 0 \Leftrightarrow abc = 0 \Leftrightarrow$ zero volume (not maximal). We look for $(x_0, y_0, z_0) \in E$ such that f reaches its maximum under the constraint $g(x, y, z) = 0$. Since $\{(x, y, z) \in E : g(x, y, z) = 0\}$ is compact (closed and bounded), and f is continuous, such a maximum (x_0, y_0, z_0) exists.

Observe, moreover, that $\nabla g(x, y, z) = (1, 2, 2) \neq \mathbf{0}$. We then search for $(x, y, z) \in E$ and $\lambda \in \mathbb{R}$ such that:

$$\begin{cases}\nabla f(x, y, z) = \lambda \nabla g(x, y, z)\\ g(x, y, z) = 0\end{cases}$$

That is, we have:

$$\begin{aligned}\frac{1}{2\sqrt{xyz}} yz &= \lambda\\ \frac{1}{2\sqrt{xyz}} xz &= 2\lambda\\ \frac{1}{2\sqrt{xyz}} xy &= 2\lambda\\ x + 2y + 2z - S &= 0\end{aligned}$$

Substituting, we get:

$$xz = 2yz \tag{1}$$
$$xy = 2yz \tag{2}$$
$$x + 2y + 2z - S = 0 \tag{3}$$

From equations (1) and (2), we obtain $y = \frac{x}{2}$ and $z = \frac{x}{2}$. Substituting into (3), we have:

$$x + x + x - S = 0 \implies x = \frac{S}{3}$$

Thus,

$$x = \frac{S}{3}, \quad y = \frac{S}{6}, \quad z = \frac{S}{6}, \quad \text{and} \quad f(x, y, z) = \sqrt{\frac{S}{3} \cdot \frac{S}{6} \cdot \frac{S}{6}} = \frac{1}{6}\sqrt{\frac{S^3}{3}} > 0.$$

For all points (x, y, z) on the boundary of E, denoted as ∂E, the function f satisfies

$f(x, y, z) = 0 < \frac{1}{6}\sqrt{\frac{S^3}{3}}$. Therefore, the final solution is given by the point $(x_0, y_0, z_0) = \left(\frac{S}{3}, \frac{S}{6}, \frac{S}{6}\right)$.

In terms of variables (a, b, c), we have:

$$a_0 = \sqrt{\frac{x_0 y_0}{z_0}} = \sqrt{\frac{S}{3}}, \quad b_0 = \sqrt{\frac{S}{3}}, \quad c_0 = \frac{1}{2}\sqrt{\frac{S}{3}}.$$

Chapter 4

Parametric Curves in $\mathbb{R}^n$

We now turn our attention to a particularly important case of vector-valued functions, where the domain is an interval of $\mathbb{R}$ and its range is a subset of $\mathbb{R}^n$ with $n \geqslant 2$, in which case there exist specific notions and terminology.

Definition 4.1 (Parametric Curve)**.** Let $n \geqslant 1$ be an integer. Given a non-empty interval $I \subseteq \mathbb{R}$, a (vector-valued) function of the from $\mathbf{f}\colon I \to \mathbb{R}^n$ is called a *parametric curve* in $\mathbb{R}^n$.

Given a parametric curve

$$\mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}, \quad t \in I,$$

the functions $f_1, \ldots, f_n$ are called the *component functions* of $\mathbf{f}$. The interval I is called the *parameter interval* of the curve and the variable t is the *parameter*. The image of f

$$\operatorname{Im} \mathbf{f} = \{\mathbf{f}(t) : t \in I\}$$

is also called the *trace* of $\mathbf{f}$. Parametric curves are often used to describe the path of a moving particle in space, where the particle's position, represented as a point in $\mathbb{R}^3$, varies with a single time-parameter t. The image of the parametric curve corresponds to the trajectory "traced" by the moving particle, thus earning the name *trace*.

Example 4.1 (Helix)**.** For $r > 0$ and $c \in \mathbb{R}$ let $f\colon \mathbb{R} \to \mathbb{R}^3$ be given by

$$f(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ ct \end{pmatrix}$$

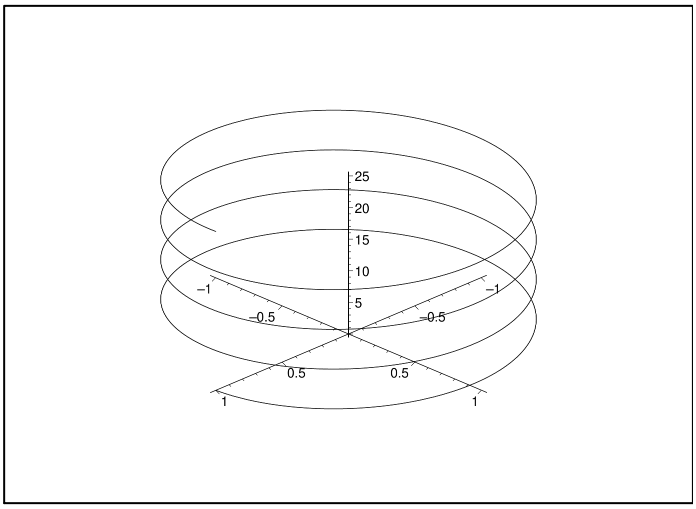


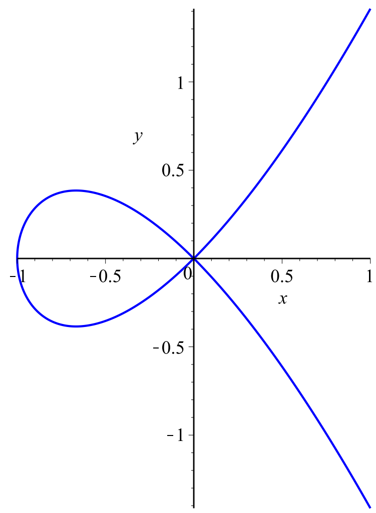
Figure 4.1: Helix

Example 4.2 (A non-injective curve)**.** Let $f\colon \mathbb{R} \to \mathbb{R}^2$ be the function

$$f(t) = \begin{pmatrix} t^2 - 1 \\ t^3 - t \end{pmatrix}$$

We have $f(-1) = f(1) = \mathbf{0}$ and

$$\operatorname{Im}(f) = f(\mathbb{R}) = \left\{(x, y) \in \mathbb{R}^2 : x^2 + x^3 = y^2\right\}$$



Example 4.3. Let us find a parametric curve whose trace represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$ (see Fig. 4.2).

Let C denote the parametric curve that we are seeking. The projection of C onto the xy-plane is the circle $x^2 + y^2 = 1,\ z = 0$. The parametrization of this circle is given by

$$x(t) = \cos(t), \quad y(t) = \sin(t), \qquad t \in [0, 2\pi).$$

From the equation of the plane, we have

$$z(t) = 2 - y(t) = 2 - \sin(t), \qquad t \in [0, 2\pi).$$

So we can write parametric function tracing the curve C as

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \qquad t \in [0, 2\pi),$$

where

$$x(t) = \cos(t), \quad y(t) = \sin(t), \quad \text{and} \quad z(t) = 2 - \sin(t).$$

The arrows on the right in Fig. 4.2 indicate the direction in which C is traced by the parametric curve $\mathbf{r}(t)$ as the parameter t ranges from 0 to 2π.

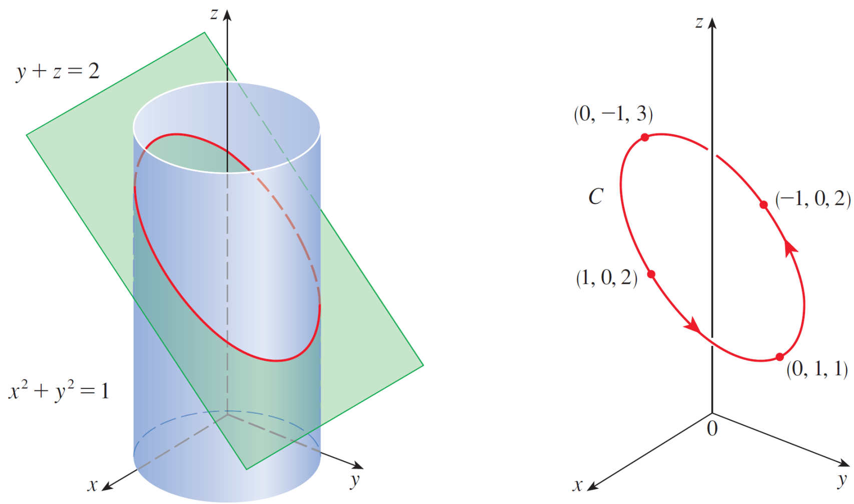


Figure 4.2

4.1 Continuity and Differentiability of Parametric Curves

Definition 4.2 (Continuity)**.** A parametric curve $\mathbf{f}\colon I \to \mathbb{R}^n$ is *continuous at* $t_0 \in I$ if and only if, for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for all $t \in I$,

$$|t - t_0| \leqslant \delta \implies \|\mathbf{f}(t) - \mathbf{f}(t_0)\|_2 \leqslant \varepsilon.$$

If $\mathbf{f}\colon I \to \mathbb{R}^n$ is continuous at every $t \in I$ then $\mathbf{f}$ is also referred to as a *path* in $\mathbb{R}^n$.

Proposition 4.1. *Suppose* $\mathbf{f}(t)\colon I \to \mathbb{R}^n$ *is a parametric curve in* $\mathbb{R}^n$ *and*

$$\mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

are its components functions. Then $\mathbf{f}(t)$ *is continuous at* t_0 *if and only if all of its component functions* $f_1(t), \ldots, f_n(t)$ *are continuous at* t_0.

Definition 4.3 (Differentiability)**.** We say that the curve $\mathbf{f}$ is *differentiable* at $t_0 \in I$ and that its *tangent vector* (or *velocity vector*) at t_0 is $\mathbf{f}'(t_0) \in \mathbb{R}^n$ if

$$\lim_{t \to t_0} \left\| \frac{\mathbf{f}(t) - \mathbf{f}(t_0)}{t - t_0} - \mathbf{f}'(t_0) \right\|_2 = 0.$$

If $\mathbf{f}$ is differentiable at t_0 and $\mathbf{f}'(t_0) \neq \mathbf{0}$ then the vector

$$\frac{1}{\|\mathbf{f}'(t_0)\|_2}\mathbf{f}'(t_0)$$

is called the *unit tangent vector*.

Proposition 4.2. *Suppose* $\mathbf{f}(t)\colon I \to \mathbb{R}^n$ *is a parametric curve in* $\mathbb{R}^n$ *and*

$$\mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

are its components functions. Then $\mathbf{f}(t)$ *is differentiable at* t_0 *if and only if all of its component functions* $f_1(t), \ldots, f_n(t)$ *are differentiable at* t_0*. In this case,*

$$\frac{d}{dt}\mathbf{f}(t_0) = \mathbf{f}'(t_0) = \begin{pmatrix} f_1'(t_0) \\ \vdots \\ f_n'(t_0) \end{pmatrix}.$$

Properties of tangent vectors: Below, we see that many of the differentiation formulas for real-valued functions have their counterparts for parametric curves.

1. **Linearity**: For all $\alpha, \beta \in \mathbb{R}$ we have

$$\frac{d}{dt}[\alpha\mathbf{u}(t) + \beta\mathbf{v}(t)] = \alpha\mathbf{u}'(t) + \beta\mathbf{v}'(t).$$

2. **Product rule for scalar products**:

$$\frac{d}{dt}[g(t)\mathbf{u}(t)] = g'(t)\mathbf{u}(t) + g(t)\mathbf{u}'(t)$$

3. **Product rule for inner products**:

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t).$$

4. **Chain rule**:

$$\frac{d}{dt}[\mathbf{u}(g(t))] = \mathbf{u}'(g(t))g'(t).$$

Definitions 4.1. Let $\mathbf{f}\colon I \to \mathbb{R}^n$ be a parametric curve with component functions $f_1, \ldots, f_n$. Let $k \geqslant 1$ be an integer. If the derivatives $f_j^{(m)}$ exist and are continuous on I for all $1 \leqslant m \leqslant k$ and all $1 \leqslant j \leqslant n$, then the curve $\mathbf{f}$ is said to be of *class* $C^k(I)$. If $\mathbf{f}$ is of class $C^k(I)$ for all $k \geqslant 1$, it is said to be of *class* $C^\infty(I)$.

Example 4.4. Consider the curve

$$x = t^3 - t, \quad y = e^{2t}, \quad z = \cos(3t).$$

Let us find the equation of the tangent line at $t = 1$.

First, we compute the Velocity Vector:

$$\mathbf{v}(t) = \left(\frac{d}{dt}(t^3 - t), \frac{d}{dt}e^{2t}, \frac{d}{dt}\cos(3t)\right) = (3t^2 - 1, 2e^{2t}, -3\sin(3t)).$$

At $t = 1$, this yields

$$\mathbf{v}(1) = (3(1)^2 - 1, 2e^2, -3\sin 3) = (2, 2e^2, -3\sin 3).$$

We also need the point of tangency, which is

$$P = (1^3 - 1, e^{2(1)}, \cos(3(1))) = (0, e^2, \cos 3).$$

We can now write the tangent line equations (in parametric form) as

$$x = 0 + 2s, \quad y = e^2 + 2e^2 s, \quad z = \cos 3 - 3\sin 3 \cdot s.$$

Thus, the tangent line at $(0, e^2, \cos 3)$ follows the direction $(2, 2e^2, -3\sin 3)$.

4.2 Motion in Space: Velocity and Acceleration

We can use vector-valued functions to represent physical quantities, such as velocity, acceleration, force, momentum, etc. For example, let the real variable t represent time elapsed from some initial time (such as $t = 0$), and suppose that an object of constant mass m is subjected to some force so that it moves in 3-dimensional space, with its position (x, y, z) at time t a function of t. That is, $x = x(t)$, $y = y(t)$, $z = z(t)$ for some real-valued functions $x(t)$, $y(t)$, $z(t)$. Call $\mathbf{r}(t) = (x(t), y(t), z(t))$ the *position vector* of the object. We can define various physical quantities associated with the object as follows:

$$\begin{aligned}
\textit{position}\colon\ \mathbf{r}(t) &= \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \\
\textit{velocity}\colon\ \mathbf{v}(t) &= \dot{\mathbf{r}}(t) = \mathbf{r}\,'(t) = \frac{d\mathbf{r}}{dt} \\
&= \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} \\
\textit{acceleration}\colon\ \mathbf{a}(t) &= \dot{\mathbf{v}}(t) = \mathbf{v}\,'(t) = \frac{d\mathbf{v}}{dt} \\
&= \ddot{\mathbf{r}}(t) = \mathbf{r}\,''(t) = \frac{d^2\mathbf{r}}{dt^2} \\
&= \begin{pmatrix} x''(t) \\ y''(t) \\ z''(t) \end{pmatrix} \\
\textit{momentum}\colon\ \mathbf{p}(t) &= m\mathbf{v}(t) \\
\textit{force}\colon\ \mathbf{F}(t) &= \dot{\mathbf{p}}(t) = \mathbf{p}\,'(t) = \frac{d\mathbf{p}}{dt} \qquad \text{(Newton's Second Law of Motion)}
\end{aligned}$$

The magnitude $\|\mathbf{v}(t)\|_2$ of the velocity vector is called the *speed* of the object. Note that since the mass m is a constant, the force equation becomes the familiar $\mathbf{F}(t) = m\mathbf{a}(t)$.

Example 4.5. Let us show that if $\|\mathbf{r}(t)\|_2 = c$ (a constant) then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

To prove this claim, we will simply use the product rule for inner products. Since

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = \|\mathbf{r}(t)\|_2^2 = c^2$$

and c^2 is a constant, we have

$$\frac{d}{dt}\Big(\mathbf{r}(t) \cdot \mathbf{r}(t)\Big) = \mathbf{0}.$$

By the product rule, the left hand side is

$$\frac{d}{dt}\Big(\mathbf{r}(t) \cdot \mathbf{r}(t)\Big) = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t).$$

Thus $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, which says that $\mathbf{r}'(t)$ and $\mathbf{r}(t)$ are orthogonal.

Example 4.6. An object with mass m that moves in a circular path with constant angular speed ω has position vector $\mathbf{r}(t) = (a\cos(\omega t), a\sin(\omega t))$. Find the force acting on the object and show that it is directed toward the origin.

To find the force, we first need to know the acceleration:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \begin{pmatrix} -a\omega\sin(\omega t) \\ a\omega\cos(\omega t) \end{pmatrix}$$
$$\mathbf{a}(t) = \mathbf{v}'(t) = \begin{pmatrix} -a\omega^2\cos(\omega t) \\ -a\omega^2\sin(\omega t) \end{pmatrix}.$$

Therefore Newton's Second Law gives the force as

$$\mathbf{F}(t) = m\mathbf{a}(t) = -m\omega^2 \begin{pmatrix} a\cos(\omega t) \\ a\sin(\omega t) \end{pmatrix}.$$

Notice that $\mathbf{F}(t) = -a\omega^2\mathbf{r}(t)$. This shows that the force acts in the direction opposite to the radius vector $\mathbf{r}(t)$ and therefore points toward the origin. Such a force is called a centripetal (center-seeking) force.

4.3 Arc Length

Definition 4.4 (Length of a Curve Arc)**.** Let there be a curve $\mathbf{f}\colon I \to \mathbb{R}^n$ of class $C^1(I)$ and let $a < b \in I$. The *arc length* of the curve $\mathbf{f}\colon [a,b] \to \mathbb{R}^n$ is defined as

$$L(\mathbf{f}) = \int_a^b \|\mathbf{f}'(t)\|_2 \,\mathrm{d}t.$$

Given that the interval $[a,b]$ is closed and bounded, $L(\mathbf{f}) < +\infty$.

Example 4.7. In $\mathbb{R}^2$, consider the circle with center $\mathbf{c} = (c_1, c_2)$ and radius $r > 0$ parameterized by

$$\mathbf{f}(\theta) = \begin{pmatrix} c_1 + r\cos(a\theta) \\ c_2 + r\sin(a\theta) \end{pmatrix} = \mathbf{c} + r\begin{pmatrix} \cos(a\theta) \\ \sin(a\theta) \end{pmatrix}, \quad \theta \in \mathbb{R}$$

where $a > 0$ is a constant. The length of the curve arc $\mathbf{f}\colon [0, 2\pi/a] \to \mathbb{R}^2$ is

$$\int_0^{2\pi/a} ra \,\mathrm{d}\theta = 2\pi r.$$

Example 4.8. Given a continuously differentiable function $g\colon I \to \mathbb{R}$, consider its parameterized graph:

$$\mathbf{f}(t) = \begin{pmatrix} t \\ g(t) \end{pmatrix}, \quad t \in I.$$

For $a < b \in I$, the arc length of the graph is therefore given by

$$\int_a^b \|\mathbf{f}'(t)\|_2 \,\mathrm{d}t = \int_a^b \sqrt{1 + (g'(t))^2} \,\mathrm{d}t.$$

Proposition 4.3 (Derivative of an Integral Depending on a Parameter)**.** *Let $a < b$ be two real numbers, I an open interval, and $f\colon [a,b] \times I \to \mathbb{R}$ a continuous function whose partial derivative with respect to the second variable exists and is continuous* *on* $[a,b]\times I$. *Then, the function* $F\colon I\to\mathbb{R}$ *defined by*

$$F(t)=\int_a^b F(x,t)\,\mathrm{d}x$$

is continuously differentiable on I *and, moreover, for every* $t\in I$, *we have:*

$$F'(t)=\int_a^b \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x.$$

Proposition 4.4. *Let* $a,b:\mathbb{R}\to\mathbb{R}$ *be of class* $C^1(\mathbb{R})$, *and* $f\colon\mathbb{R}^2\to\mathbb{R}$ *of class* $C^1(\mathbb{R}^2)$, *and define* $F(t)$ *by*

$$F(t)=\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x.$$

Then F *is continuously differentiable on* $\mathbb{R}$ *and*

$$F'(t)=F(b(t),t)\cdot b'(t)-F(a(t),t)\cdot a'(t)+\int_{a(t)}^{b(t)}\frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x.$$

Example 4.9. 1) Given $F(t)=\int_0^\pi \frac{\sin(tx)}{x}\,dx$, let us calculate $F'\left(\frac{1}{4}\right)$. First, note that $f(x,t)$ is of class $C^1(\mathbb{R}^2)$ (which needs verification!). So it follows that

$$\begin{aligned} F'(t) &= \int_0^\pi \frac{\cos(tx)x}{x}\,dx F'(t)=\int_0^\pi \frac{\cos(tx)x}{x}\,dx \\ &= \left[\frac{1}{t}\sin(tx)\right]_{x=0}^{x=\pi} \\ &= \frac{1}{t}\sin(\pi t). \end{aligned}$$

Hence, we have

$$F'\left(\frac{1}{4}\right)=4\sin\left(\pi\cdot\frac{1}{4}\right)=4\cdot\frac{\sqrt{2}}{2}=2\sqrt{2}$$

2) Next let us find $F'\left(\frac{1}{4}\right)$ when $F(t)=\int_0^{t^2}\frac{\sin(tx)}{x}\,dx$. We have

$$\begin{aligned} F'(t) &= \frac{1}{t^2}\sin\Big(t\cdot t^2\Big)\cdot(2t)+\left[\frac{1}{t}\sin(tx)\right]_{x=0}^{x=t^2} \\ &= \frac{2}{t}\sin\Big(t^3\Big)+\frac{1}{t}\sin\Big(t^3\Big)=\frac{3}{t}\sin\Big(t^3\Big). \end{aligned}$$

This now gives

$$F'\left(\frac{1}{4}\right)=12\sin\left(\frac{1}{64}\right).$$

Chapter 5

Vector Calculus

In this chapter, we study the calculus of multivariable vector-valued functions and vector fields. These are functions that assign vectors to points in space.

5.1 Functions with values in $\mathbb{R}^m$

A *vector-valued function*, sometimes also referred to as a *vector function*, is a mathematical function of one or more variables whose output values are multidimensional vectors. In other words, it is a function of the form $\mathbf{f}\colon E \to \mathbb{R}^m$ whose domain $\operatorname{dom}(\mathbf{f}) = E$ is a subset of $\mathbb{R}^n$ and its image $\operatorname{im}(\mathbf{f}) = \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in E\}$ is a subset of $\mathbb{R}^m$. Every vector-valued function $\mathbf{f}\colon E \to \mathbb{R}^m$ can be viewed as an m-tuple of real-valued functions,

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^m,$$

where $f_1, \ldots, f_m\colon E \to \mathbb{R}$ are called the *component functions* of $\mathbf{f}$.

We have already encountered several types of vector-valued functions in this course. For example, in Chapter 4 we discussed vector-valued functions of the form $f\colon \mathbb{R} \to \mathbb{R}^m$, called *parametric curves*. Also, in Section 3.1 we introduced the gradient vector $\nabla f(\mathbf{x})$ and in Section 3.6 the Hessian matrix $\operatorname{Hess}(f)(\mathbf{x})$, which are both examples of vector-valued functions. Indeed, $\nabla f\colon \mathbb{R}^n \to \mathbb{R}^{1\times n}$ is a vector-valued function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^{1\times n} \cong \mathbb{R}^n$ (where we can identify the space of n-dimensional row vectors $\mathbb{R}^{1\times n}$ with the space of n-dimensional column vectors $\mathbb{R}^n$), and the Hessian matrix $\operatorname{Hess}(f)\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a vector-valued function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ (where we can identify the space of $n \times n$ matrices with the euclidean vector space $\mathbb{R}^{n^2}$ of dimension n^2).

5.2 Limits and Continuity of Vector-valued Functions

The concepts of limits and continuity can be extended to functions $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ in a straightforward manner.

Definition 5.1. Let $\mathbf{f}\colon E \to \mathbb{R}^m$ with $E \subseteq \mathbb{R}^n$. We say that $\mathbf{f}$ is *defined in a neighborhood of* $\mathbf{a}$ if $\mathbf{a}$ is an interior point of $E \cup \{\mathbf{a}\}$.

Definition 5.2 (Limit of a function)**.** Let $\mathbf{a}$ be a point in $\mathbb{R}^n$, and let $\mathbf{f}\colon E \to \mathbb{R}^m$ with $E \subseteq \mathbb{R}^n$ be a vector-valued function defined in a neighborhood of $\mathbf{a}$. Then we say that the *limit* of $\mathbf{f}(\mathbf{x})$ equals $\mathbf{L} \in \mathbb{R}^m$ as $\mathbf{x}$ approaches $\mathbf{a}$, written as

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}, \tag{5.1}$$

if given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{a}\|_2 < \delta \quad \Longrightarrow \quad \|\mathbf{f}(\mathbf{x}) - \mathbf{L}\|_2 < \varepsilon.$$

It is sufficient to check component functions for limits of vector-valued functions, as evidenced by the next proposition, because the convergence of each component function guarantees the convergence of the vector-valued function as a whole.

Proposition 5.1. *Suppose* $\mathbf{f}\colon E \to \mathbb{R}^m$ *is a vector-valued function defined in a neighborhood of* $\mathbf{a} \in \mathbb{R}^n$*. If*

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} \qquad \text{and} \qquad \mathbf{L} = \begin{pmatrix} L_1 \\ \vdots \\ L_m \end{pmatrix}$$

then $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$ *if and only if* $\lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x}) = L_i$ *for all* $1 \leqslant i \leqslant m$*.*

Definition 5.3 (Continuity at a point)**.** Let $\mathbf{a}$ be an interior point of E. A function $\mathbf{f}\colon E \to \mathbb{R}^m$ is *continuous at* $\mathbf{a}$ if and only if, for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for all $\mathbf{x} \in E$,

$$\|\mathbf{x} - \mathbf{a}\|_2 \leqslant \delta \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\|_2 \leqslant \varepsilon.$$

Continuity for vector-valued functions is ensured if and only if all component functions are continuous, akin to the situation with limits. This allows known principles about continuity of real-valued functions to generalize directly to vector-valued functions, as the following proposition demonstrates

Proposition 5.2. *Suppose*

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} : E \to \mathbb{R}^m$$

is a vector-valued function and $\mathbf{a}$ *an interior point of* E*. The following are equivalent:*

(i) $\mathbf{f}(\mathbf{x})$ *is continuous at* $\mathbf{a}$*;*
(ii) $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$*;*
(iii) for every sequence $(\mathbf{a}_k)_{k\in\mathbb{N}}$ *of elements of* E *we have*

$$\lim_{k\to+\infty} \mathbf{a}_k = \mathbf{a} \quad \Longrightarrow \quad \lim_{k\to+\infty} \mathbf{f}(\mathbf{a}_k) = \mathbf{f}(\mathbf{a});$$

(iv) $f_i(\mathbf{x})$ *is continuous at* $\mathbf{a}$ *for all* $1 \leqslant i \leqslant m$*.*

Remark 5.1. Now that we understand what it means for a vector-valued function to be continuous, we can revisit the definition of the class of C^1 and C^2 functions given in Chapter 3. Let $E \subseteq \mathbb{R}^n$ be an open set and let $f\colon E \to \mathbb{R}$ be a real-valued function in n variables. In light of Proposition 5.2, we see that f is of class $C^1(E)$, as specified in Definition 3.7, if and only if the gradient vector $\nabla f\colon E \to \mathbb{R}^n$ is continuous as a vector-valued function. Similarly, f is of class $C^2(E)$, as specified in Definition 3.11, if and only if its Hessian matrix $\operatorname{Hess}(f)\colon E \to \mathbb{R}^{n\times n}$ is a continuous vector-valued function from E to $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$.

5.3 Partial and Directional Derivatives of Vector-valued Functions

The partial derivatives of a multivariable real-valued function are real numbers (see Definition 3.1). In analogy, the partial derivatives of a multivariable vector-valued function are vectors.

Definition 5.4 (Partial derivatives)**.** Let $E \subseteq \mathbb{R}^n$ be open and $\mathbf{f}\colon E \to \mathbb{R}^m$ a vector-valued function in the variables $x_1, \ldots, x_n$. Then $\mathbf{f}$ has a *partial derivative* at the point $a \in E$ with respect to the variable x_j if each of its component functions $f_1, \ldots, f_m$ has a partial derivative at the point a with respect to the variable x_j. In this case, we denote the partial derivative of $\mathbf{f}$ with respect to the variable x_j as an m-dimensional column vector:

$$\frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{a}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_j}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_j}(\mathbf{a}) \\ \vdots \\ \frac{\partial f_m}{\partial x_j}(\mathbf{a}) \end{pmatrix}.$$

Definition 5.5 (Jacobian matrix)**.** Let $E \subseteq \mathbb{R}^n$ be an open set, let $\mathbf{f}\colon E \to \mathbb{R}^m$ be a function and suppose all partial derivatives $\frac{\partial \mathbf{f}}{\partial x_1}(\mathbf{a}), \ldots, \frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{a})$ of $\mathbf{f}$ at the point $\mathbf{a} \in E$

exist. The matrix

$$D\mathbf{f}(\mathbf{a}) = \mathbf{J_f}(\mathbf{a}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \frac{\partial f_m}{\partial x_2}(\mathbf{a}) & \ldots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{pmatrix}$$

is called the *Jacobian matrix* or the *Jacobian* of $\mathbf{f}$ at the point $\mathbf{a}$. It is an $m \times n$ matrix, i.e., it has m rows and n columns. The columns correspond to the partial derivatives $\frac{\partial \mathbf{f}}{\partial x_1}(\mathbf{a}), \ldots, \frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{a})$, whereas the rows correspond to the gradients of the component functions $\nabla f_1(\mathbf{a}), \ldots, \nabla f_m(\mathbf{a})$.

When $m = n$, the Jacobian matrix is a square matrix and its determinant

$$\frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)}(\mathbf{a}) = \det \mathbf{J_f}(\mathbf{a}) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \ldots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \ldots & \frac{\partial f_2}{\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{a}) & \frac{\partial f_n}{\partial x_2}(\mathbf{a}) & \ldots & \frac{\partial f_n}{\partial x_n}(\mathbf{a}) \end{vmatrix}$$

is denoted as $\frac{D(f_1,\ldots,f_n)}{D(x_1,\ldots,x_n)}(\mathbf{a})$ and called the *Jacobian determinant* of $\mathbf{f}$ at the point $\mathbf{a}$.

Example 5.1. If $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbf{f}(x, y) = (xy, x + y)$, then $\mathbf{J_f}(1, 2)$ can be calculated as

$$\mathbf{J_f}(x, y) = \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} \qquad \text{and hence} \qquad \mathbf{J_f}(1, 2) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Example 5.2. Suppose $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$.

- if $n = m = 1$ then $\mathbf{f}$ is a real-valued single-variable function and its Jacobian $\mathbf{J_f}(\mathbf{a})$, which is 1×1 matrix, coincides with the derivative $\mathbf{f}'(\mathbf{a})$.
- if $m = 1$ and n is arbitrary then $\mathbf{f}$ is a real-valued function in n variables and its Jacobian $\mathbf{J_f}(\mathbf{a})$, which is a $1 \times n$ matrix, is the same as the gradient $\nabla \mathbf{f}(\mathbf{a})$.
- if $n = 1$ and m is arbitrary then $\mathbf{f}$ is a parametric curve in $\mathbb{R}^m$ and its Jacobian $\mathbf{J_f}(\mathbf{a})$ is the same as the tangent vector $\mathbf{f}'(\mathbf{a})$.

Definition 5.6 (Directional derivatives)**.** Let $E \subseteq \mathbb{R}^n$ be open and $\mathbf{f}\colon E \to \mathbb{R}^m$ a vector-valued function. Then $\mathbf{f}$ has a *directional derivative* along the vector $\mathbf{v} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ at the point $a \in E$ if each of its component functions $f_1, \ldots, f_m$ has a directional derivative along $\mathbf{v}$ at the point a. In this case, we denote the directional derivative of $\mathbf{f}$ along $\mathbf{v}$ as an m-dimensional column vector:

$$\nabla_{\mathbf{v}} \mathbf{f}(\mathbf{a}) = \begin{pmatrix} \nabla_{\mathbf{v}} f_1(\mathbf{a}) \\ \nabla_{\mathbf{v}} f_2(\mathbf{a}) \\ \vdots \\ \nabla_{\mathbf{v}} f_m(\mathbf{a}) \end{pmatrix}.$$

When $\|\mathbf{v}\|_2 = 1$, it is also called the *derivative in the direction* $\mathbf{v}$.

5.4 Differentiability of Vector-valued Functions

We have already learned what it means for functions $\mathbb{R}^n \to \mathbb{R}$ to be differentiable (see Definition 3.4), and what it means for functions $\mathbb{R} \to \mathbb{R}^m$ to be differentiable (see Definition 4.3). The following definition encompasses both of these cases and provides the general framework to discuss differentiability for multivariable vector-valued functions.

Definition 5.7 (Differentiability at a point)**.** Let E be a non-empty subset of $\mathbb{R}^n$. A function $\mathbf{f}\colon E \to \mathbb{R}$ is *differentiable* at the point $\mathbf{a} \in E$ if there exists a linear map $\mathbf{L_a}\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\left\|\mathbf{f}\left(\mathbf{a}+\mathbf{h}\right)-\mathbf{f}(\mathbf{a})-\mathbf{L_a}(\mathbf{h})\right\|_2}{\|\mathbf{h}\|_2} = 0.$$

In this case, the linear map $\mathbf{L_a}\colon \mathbb{R}^n \to \mathbb{R}^m$ is called the **differential** of $\mathbf{f}$ at the point $\mathbf{a}$.

Proposition 5.3. *Let $\mathbf{f}\colon E \to \mathbb{R}^m$ and $\mathbf{a} \in E \subseteq \mathbb{R}^n$. Then $\mathbf{f}$ is differentiable at $\mathbf{a}$ if and only if all its component functions $f_1, \ldots, f_m$ are differentiable at $\mathbf{a}$.*

Theorem 5.1 (Fundamental theorem)**.** *Suppose $\mathbf{f}\colon E \to \mathbb{R}^m$ is differentiable at a point $\mathbf{a} \in E$. Then the following statements hold.*

(i) *$\mathbf{f}$ is continuous at $\mathbf{a}$.*

(ii) *All partial derivatives of $\mathbf{f}$ at the point $\mathbf{a}$ exist, the Jacobian matrix $\mathbf{J_f}(\mathbf{a})$ of $\mathbf{f}$ at the point $\mathbf{a}$ exists, and the differential $\mathbf{L_a}\colon \mathbb{R}^n \to \mathbb{R}^m$ of $\mathbf{f}$ at the point $\mathbf{a}$ is the same as matrix multiplication with the Jacobian matrix, i.e.,*

$$\mathbf{L_a}(\mathbf{v}) = \mathbf{J_f}(\mathbf{a}) \cdot \mathbf{v}, \qquad \forall \mathbf{v} \in \mathbb{R}^n.$$

(iii) *All directional derivatives of $\mathbf{f}$ at the point $\mathbf{a}$ exist and are given by*

$$\nabla_{\mathbf{v}}\mathbf{f}(\mathbf{a}) = \mathbf{J_f}(\mathbf{a}) \cdot \mathbf{v}, \qquad \forall \mathbf{v} \in \mathbb{R}^n.$$

(iv) *For all $\mathbf{x} \in E$ we have*

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + \mathbf{J_f}(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a}) + \mathbf{r}_1(\mathbf{x}),$$

where $\mathbf{r}_1$ is an "error" term satisfying

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{r}_1(\mathbf{x})\|_2}{\|\mathbf{x}-\mathbf{a}\|_2} = 0.$$

The function

$$\mathbf{t}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + \mathbf{J_f}(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a})$$

*is called the **linearization** (or **linear approximation**) of $\mathbf{f}$ at the point $\mathbf{a}$.*

Among other things, the above theorem implies that if $\mathbf{f}$ is differentiable at $\mathbf{a}$ then

$$\nabla_{\mathbf{v}}\mathbf{f}(\mathbf{a}) = \begin{pmatrix} \nabla_{\mathbf{v}} f_1(\mathbf{a}) \\ \nabla_{\mathbf{v}} f_2(\mathbf{a}) \\ \vdots \\ \nabla_{\mathbf{v}} f_m(\mathbf{a}) \end{pmatrix} = \begin{pmatrix} \langle \nabla f_1(\mathbf{a}), \mathbf{v}\rangle \\ \langle \nabla f_2(\mathbf{a}), \mathbf{v}\rangle \\ \vdots \\ \langle \nabla f_m(\mathbf{a}), \mathbf{v}\rangle \end{pmatrix} = \mathbf{J_f}(\mathbf{a}) \cdot \mathbf{v} = \sum_{j=1}^{n} v_j \frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{a})$$

for all $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n \backslash \{\mathbf{0}\}$.

Example 5.3. Let $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $\mathbf{f}(1,2) = (3,-1)$ and $\mathbf{J_f}(1,2) = \left(\begin{smallmatrix} 1 & -1 \\ -3 & 0 \end{smallmatrix}\right)$. We can use this limited amount of information to approximate $\mathbf{f}(1.1, 1.8)$. Indeed, the linear approximation of $\mathbf{f}$ at the point $(1,2)$ is

$$\begin{aligned} \mathbf{t}(x,y) &= \mathbf{f}(1,2) + \mathbf{J_f}(1,2) \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) \\ &= \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -3 & 0 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} \\ &= \begin{pmatrix} x - y + 4 \\ 2 - 3x \end{pmatrix}. \end{aligned}$$

Thus, as an approximation of $\mathbf{f}(1.1, 1.8)$ we obtain

$$\mathbf{f}(1.1, 1.8) \approx \mathbf{t}(1.1, 1.8) = \begin{pmatrix} 3.3 \\ -1.3 \end{pmatrix}.$$

5.5 Vector-Valued Functions of Class C^1

Definition 5.8. Let $E \subseteq \mathbb{R}^n$ be an open set and let $\mathbf{f}\colon E \to \mathbb{R}^m$ be a function. We say that $\mathbf{f}$ is of class $C^1(E)$ if all partial derivatives $\frac{\partial \mathbf{f}}{\partial x_1}(\mathbf{a}), \ldots, \frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{a})$ of $\mathbf{f}$ exist and are continuous at every point $\mathbf{a} \in E$.

It follows from the definition that $\mathbf{f}\colon E \to \mathbb{R}^m$ is of class $C^1(E)$ if and only if the Jacobian matrix $\mathbf{J_f}(\mathbf{a})$ exists at every point $\mathbf{a} \in E$ and the map $\mathbf{J_f}\colon E \to \mathbb{R}^{m\times n}$ is a continuous function. So, continuity of the Jacobian matrix is the multivariable analogue of continuous differentiability for vector-valued functions.

Proposition 5.4. *Let $E \subseteq \mathbb{R}^n$ be an open set, let $\mathbf{f}\colon E \to \mathbb{R}^m$ be a function, and let $f_1, \ldots, f_m\colon E \to \mathbb{R}$ be its component functions. Then $\mathbf{f}$ is of class $C^1(E)$ if and only if all its component functions $f_1, \ldots, f_m$ are of class $C^1(E)$.*

Recall that real-valued functions of class C^1 are always differentiable (cf. Proposition 3.1). The next corollary, which follows by combining Proposition 3.1, Proposition 5.4 and Proposition 5.3, asserts that the same is true for vector-valued functions.

Corollary 5.1. *Let $E \subseteq \mathbb{R}^n$ be an open set, let $\mathbf{f}\colon E \to \mathbb{R}^m$ be a function. If $\mathbf{f}$ is of class $C^1(E)$ then $\mathbf{f}$ is differetniable at every point in E.*

5.6 The Chain Rule

The goal of this section is to introduce the chain rule for multivariable vector-valued functions. As motivation, let us first recall the chain rule for single-variable functions as you have learned it in Analysis I: If $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are functions such that g is differentiable at a point a and f is differentiable at the point $g(a)$, then the composition $f \circ g$ is differentiable at the point a, and its derivative is given by

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

This expresses that the rate of change of $f \circ g$ at a is the product of the rate of change of f at $g(a)$ and the rate of change of g at a.

The following theorem is the appropriate generalization of the chain rule to higher dimensions.

Theorem 5.2 (Chain Rule)**.** *Suppose we are given an open subset $A \subseteq \mathbb{R}^n$, a function $\mathbf{g}\colon A \to \mathbb{R}^p$, an open subset $B \subseteq \mathbb{R}^p$ with $\mathbf{g}(A) \subseteq B$, and a function $\mathbf{f}\colon B \to \mathbb{R}^q$. Therefore, the composite function $\mathbf{f} \circ \mathbf{g}\colon A \to \mathbb{R}^q$ is well-defined. If $\mathbf{a} \in A$ and $\mathbf{g}(\mathbf{a}) \in B$ such that $\mathbf{g}$ is differentiable at $\mathbf{a}$ and $\mathbf{f}$ is differentiable at $\mathbf{g}(\mathbf{a})$, then $\mathbf{f} \circ \mathbf{g}$ is differentiable at $\mathbf{a}$ and and the Jacobian matrix $\mathbf{J}_{\mathbf{f}\circ\mathbf{g}}(\mathbf{a}) \in \mathbb{R}^{q\times n}$ is the matrix product of the Jacobian matrices $\mathbf{J}_{\mathbf{f}}(\mathbf{g}(\mathbf{a})) \in \mathbb{R}^{q\times p}(\mathbb{R})$ and $\mathbf{J}_{\mathbf{g}}(\mathbf{a}) \in \mathbb{R}^{p\times n}$:*

$$\mathbf{J}_{\mathbf{f}\circ\mathbf{g}}(\mathbf{a}) = \mathbf{J}_{\mathbf{f}}(\mathbf{g}(\mathbf{a})) \cdot \mathbf{J}_{\mathbf{g}}(\mathbf{a}).$$

Furthermore, if $n = p = q$, then the following relationship for the Jacobian determinants is obtained:

$$|\mathbf{J}_{\mathbf{f}\circ\mathbf{g}}(\mathbf{a})| = |\mathbf{J}_{\mathbf{f}}(\mathbf{g}(\mathbf{a}))| \cdot |\mathbf{J}_{\mathbf{g}}(\mathbf{a})|.$$

Example 5.4. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{g}\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $\mathbf{g}(x, y) = (x^2y,\ x - y)$ and $h = f \circ \mathbf{g}$. Let us find $\frac{\partial h}{\partial x}(1, 2)$, assuming that $\frac{\partial f}{\partial x}(2, -1) = 3$ and $\frac{\partial f}{\partial y}(2, -1) = -2$. First, the Jacobian matrix of the function $\mathbf{g}(x, y) = (x^2y, x - y)$ is

$$\mathbf{J}_{\mathbf{g}}(x, y) = \begin{pmatrix} 2xy & x^2 \\ 1 & -1 \end{pmatrix}.$$

Therefore, the Jacobian at the point $(x, y) = (1, 2)$ equals

$$\mathbf{J}_{\mathbf{g}}(1, 2) = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}.$$

Also, we know that

$$g(1, 2) = (2, -1) \qquad \text{and} \qquad \nabla f(2, -1) = (3, -2).$$

So, it follows form the chain rule that

$$\nabla h(1, 2) = \nabla f(2, -1) \cdot \mathbf{J}_{\mathbf{g}}(1, 2) = (3,\ -2) \cdot \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} = (10,\ 5).$$

We deduce that $\frac{\partial h}{\partial x}(1,2) = 10$.

5.7 Method of Lagrange Multipliers – multiple constraints

In Lagrange multipliers for a single constraint, we introduce a new variable, usually denoted as λ, called the Lagrange multiplier, to determine when the gradient of the objective function is parallel to the gradient of the constraint function. When dealing with multiple constraints, each constraint adds a new term with its respective Lagrange multiplier. So, if we have m constraints then we introduce m Lagrange multipliers, usually denoted as $\lambda_1, \dots, \lambda_m$.

Theorem 5.3 (Lagrange Multiplier Theorem – multiple constraints)**.** *Consider an open set $E \subseteq \mathbb{R}^n$, functions $f, g_1, \dots, g_m \colon E \to \mathbb{R}$ of class $C^1(E)$ and constants $c_1, \dots, c_m \in \mathbb{R}$. If the function $f(\mathbf{x})$ achieves a local extreme value subject to the constraints $g_1(\mathbf{x}) = c_1, \dots, g_m(\mathbf{x}) = c_m$ at a point $\mathbf{a} \in E$ and additionally the vectors $\nabla g_1(\mathbf{a}), \dots, \nabla g_m(\mathbf{a})$ are linearly independent then there must exist scalar numbers $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that $\nabla f(\mathbf{a}) = \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{a})$. The numbers λ_i are called the* Lagrange multipliers.

Example 5.5. The planes $x + y - z = 3$ and $x - y + z = -1$ intersect in a line. Find the point on this line that is closest to the origin.

In other words, we have to minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints

1. $g_1(x, y, z) = x + y - z = 3$,
2. $g_2(x, y, z) = x - y + z = -1$.

To solve this problem using Lagrange multipliers, we need to take the partial derivatives. We get

$$\nabla f(x, y, z) = (2x, 2y, 2z)$$

and

$$\nabla g_1(x, y, z) = (1, 1, -1) \qquad \nabla g_2(x, y, z) = (1, -1, 1).$$

It is important do not forget checking linear dependence: The vectors $(1, 1, -1)$ and $(1, -1, 1)$ are linearly independent. So we can use the method of Lagrange multipliers and obtain

$$\nabla f(x, y, z) = \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z)$$

which is equivalent to

$$(2x, 2y, 2z) = (\lambda_1 + \lambda_2, \lambda_1 - \lambda_2, \lambda_2 - \lambda_1).$$

This leaves us with five equations:

$$\begin{aligned} 2x &= \lambda_1 + \lambda_2, \\ 2y &= \lambda_1 - \lambda_2, \\ 2z &= \lambda_2 - \lambda_1, \\ x + y - z &= 3, \\ x - y + z &= -1. \end{aligned}$$

Solving these equations simultaneously will give us the values of x, y, z, λ_1, and λ_2 at the critical points, which yield potential solutions to our optimization problem. In particular, using some basic algebra, we obtain the solution

$$(x, y, z, \lambda_1, \lambda_2) = (1, 1, -1, 2, 0).$$

So the point that lies on both planes simultaneously and is closest to the origin is $(1, 1, -1)$.

5.8 Finding Global Extreme Values on compact sets defined by inequalities.

In Section 3.11 we have seen a "3-step recipe" of how to find the global extreme values of a functions of class C^1 on a compact set. If the compact set is given by an inequality, we can further refine this recipe as follows:

Finding Global Extreme Values on compact sets defined by inequalities. Let $D \subseteq \mathbb{R}^n$ be open and let $g\colon E \to \mathbb{R}$ be of class $C^1(E)$, where $E := \{\mathbf{x} \in D : g(\mathbf{x}) \leqslant 0\} \subseteq D$. Suppose, moreover, that E is non-empty and compact (i.e., closed and bounded). Let $f\colon D \to \mathbb{R}$ be of class $C^1(D)$. In order to find the points where f attains a global maximum or minimum in E, it suffices to follow these steps:

1. Determine the stationary points of f in $\{\mathbf{x} \in D : g(\mathbf{x}) < 0\}$.
2. Determine the points $\mathbf{x} \in D$ such that $g(\mathbf{x}) = 0$ and $\nabla g(\mathbf{x}) = \mathbf{0}$.
3. Determine the points $\mathbf{x} \in D$ such that $g(\mathbf{x}) = 0$, $\nabla g(\mathbf{x}) \neq \mathbf{0}$, and there exists $\lambda \in \mathbb{R}$ that satisfies $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$.
4. Evaluate f at the points identified in steps 1, 2, and 3 above and compare their corresponding values.

Remark 5.2.

- Do not forget to check 2!
- Do not forget to check that E is compact! Otherwise, we cannot be sure that f attains its maximum and/or minimum.

Example 5.6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(\mathbf{x}) = x_1 \cdot \ldots \cdot x_n = \prod_{i=1}^n x_i$. Find the extrema of the restriction of f to the closed unit ball.

Here, $g(\mathbf{x}) = \sum_{i=1}^n x_i^2 - 1$ for $\mathbf{x} \in \mathbb{R}^n$. The functions f and g are continuously differentiable, and the set $E = \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leqslant 0\}$ is compact. Suppose first that $g(\mathbf{x}) < 0$, which is equivalent to $\|\mathbf{x}\|_2 < 1$. In this region, any point with at least two coordinates equal to zero is a stationary point of f, since the gradient of f vanishes there.

We want to find $\mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that:

$$\begin{cases} \|\mathbf{x}\|_2 = 1, \\ x_2 \cdot \ldots \cdot x_n = \lambda 2x_1, \\ x_1 x_3 \cdots x_n = \lambda 2 x_2, \\ \vdots \\ x_1 \cdots x_{n-1} = \lambda 2 x_n, \end{cases} \Leftrightarrow \begin{cases} x_1 x_2 \cdot \ldots \cdot x_n = 2\lambda x_1^2, \\ x_1 x_2 \cdot \ldots \cdot x_n = 2\lambda x_2^2, \\ \vdots \\ x_1 x_2 \cdots x_n = 2\lambda x_n^2. \end{cases}$$

Adding up all these equations, we obtain:

$$n f(\mathbf{x}) = 2\lambda \|\mathbf{x}\|_2^2 = 2\lambda.$$

Therefore, $\lambda = \frac{nf(\mathbf{x})}{2}$.

If $f(\mathbf{x}) = 0$, then $\lambda = 0$ and $\nabla f(\mathbf{x}) = \mathbf{0}$, which implies that at least two coordinates of $\mathbf{x}$ must be zero.

If $f(\mathbf{x}) \neq 0$, then we have:

$$f(\mathbf{x}) = 2\lambda x_1^2 = 2\lambda x_2^2 = \ldots = 2\lambda x_n^2.$$

This leads to $x_i = \pm\frac{1}{\sqrt{n}}$ for all $i \in \{1, \ldots, n\}$, and hence $\mathbf{x} = \underbrace{\left(\pm\frac{1}{\sqrt{n}}, \ldots, \pm\frac{1}{\sqrt{n}}\right)}_{2^n \text{ possibilities}}$.

Comparison:

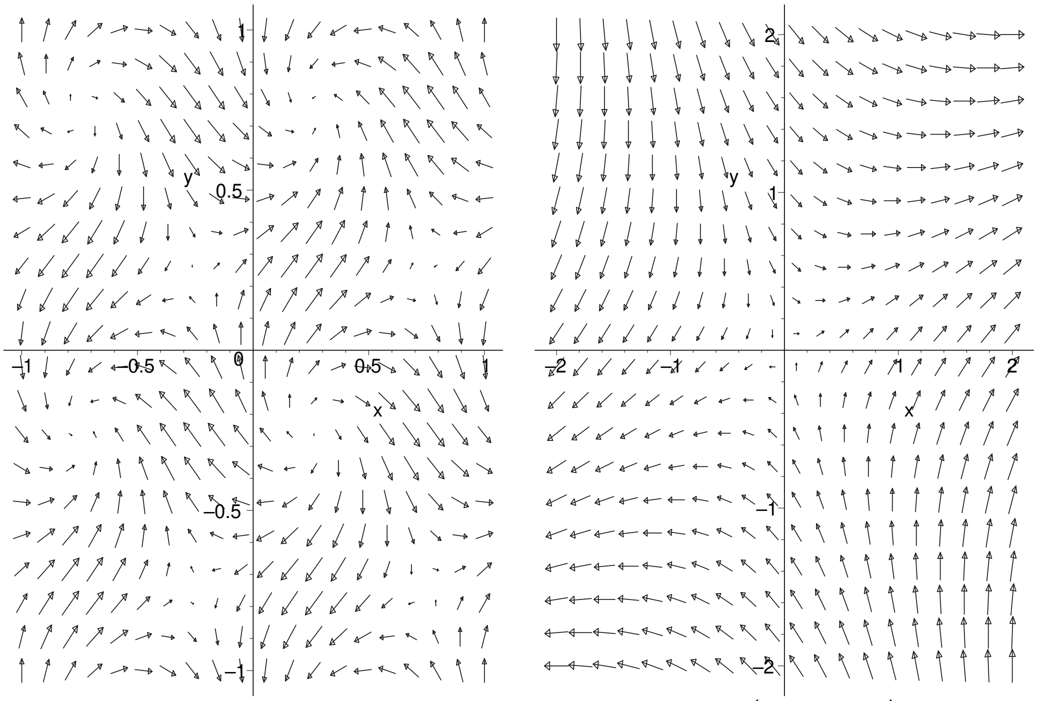
- If $\exists i \neq j$ s.t. $x_i = x_j = 0 \Rightarrow f(\mathbf{x}) = 0$.
- If $\mathbf{x} = \left(\pm\frac{1}{\sqrt{n}}, \ldots, \pm\frac{1}{\sqrt{n}}\right)$, then $f(\mathbf{x}) = \pm\left(\frac{1}{\sqrt{n}}\right)^n = \pm n^{-\frac{n}{2}}$.

Answer: If $\mathbf{x} = \left(\pm\frac{1}{\sqrt{n}}, \ldots, \pm\frac{1}{\sqrt{n}}\right)$ with an even (respectively, odd) number of negative signs, then f attains its maximum (respectively, minimum) in E, with value $n^{-\frac{n}{2}}$ (respectively, $-n^{-\frac{n}{2}}$).

5.9 Vector Fields $\mathbb{R}^n \to \mathbb{R}^n$

In general, a multivariable vector-valued function describes a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, where n represents the input dimensions and m denotes the output dimensions. If the number of input dimensions equals the number of output dimensions (i.e., $n = m$), then such a function has called a *vector field*. Vector fields show up often in many natural situations and find important applications. For example, in physics they describe magnetic and electric fields or the velocity field of a fluid. Coordinate changes are also applications $\mathbb{R}^n \to \mathbb{R}^n$.

Graphic representation. Let $U \subseteq \mathbb{R}^n$. A vector field $\mathbf{v}\colon U \to \mathbb{R}^n$ is represented graphically by an arrow (i.e. a vector) attached at each point $\mathbf{x} \in \mathbb{R}^n$.



$$(x,y) \mapsto \begin{pmatrix} \sin(\pi(x+y)) \\ \cos(\pi(x-y)) \end{pmatrix} \qquad (x,y) \mapsto \begin{pmatrix} \frac{x+y}{\sqrt[4]{4(x^2+y^2)}} \\ \frac{x-y}{\sqrt[4]{4(x^2+y^2)}} \end{pmatrix}$$

Figure 5.1: Graphic representation of vector fields.

Example 5.7. Newton's Law of force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{GmM}{r^2}$$

where r is the distance between the objects and G is the gravitational constant. (This is an example of an inverse square law.) Let's assume that the object with mass M is located at the origin in $\mathbb{R}^3$. For instance, M could be the mass of the earth and the origin would be at its center. Let the position vector of the object with mass m be $\mathbf{x} = (x, y, z)$. Then $r = \|\mathbf{x}\|_2$, so $r^2 = \|\mathbf{x}\|_2^2$. The gravitational force exerted on this second object acts toward the origin (compare with Example 4.6), and the unit vector in this direction is

$$-\frac{\mathbf{x}}{\|\mathbf{x}\|_2}.$$

Therefore, the gravitational force acting on the object at $\mathbf{x} = (x, y, z)$ is

$$\mathbf{F}(\mathbf{x}) = -\frac{GmM}{\|\mathbf{x}\|_2^3}\mathbf{x} \tag{5.2}$$

(Physicists often use the notation $\mathbf{r}$ instead of $\mathbf{x}$ for the position vector, so you may see (5.2) written in the form $\mathbf{F} = -\left(\frac{GmM}{r^3}\right)\mathbf{r}$.)

An example of a vector field is the gravitational field, because it associates a vector, the force $\mathbf{F}(\mathbf{x})$, with every point $\mathbf{x}$ in space.

Equation (5.2) is a compact way of writing the gravitational field, but we can also write it in terms of its component functions by using the facts that $\mathbf{x} = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$ and $\|\mathbf{x}\|_2 = \sqrt{x^2 + y^2 + z^2}$

$$\mathbf{F}(x, y, z) = \frac{-GmMx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{e_1} + \frac{-GmMy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{e_2} + \frac{-GmMz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{e_3}$$

The gravitational field $\mathbf{F}$ is pictured in Fig. 5.2.

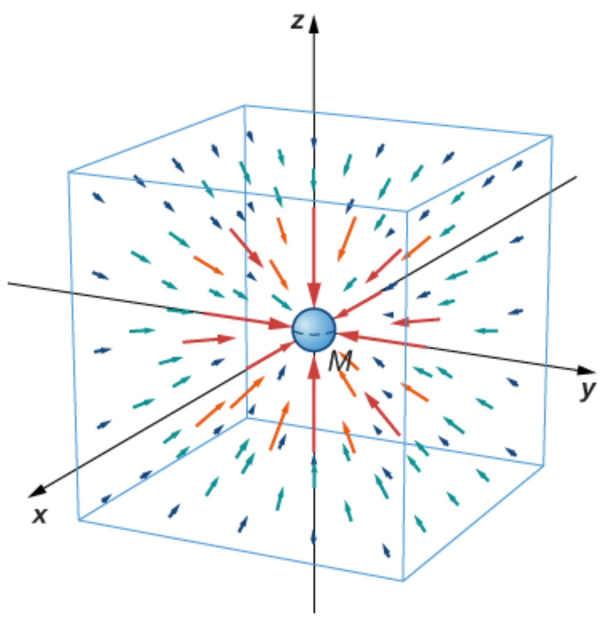


Figure 5.2: Depiction of a gravitational vector field.

Example 5.8. Consider an electric charge Q located at the origin $(0, 0, 0)$. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a point charge q located at the position $\mathbf{x} = (x, y, z)$ is

$$\mathbf{F}(\mathbf{x}) = \frac{\varepsilon qQ}{\|\mathbf{x}\|_2^3}\mathbf{x} \tag{5.3}$$

where ε is a constant (that depends on the units used). For like charges, we have $qQ > 0$ and the force is repulsive; for unlike charges, we have $qQ < 0$ and the force is attractive. Notice the similarity between (5.2) and (5.3). Both vector fields are examples of so-called *force fields*.

Instead of considering the electric force $\mathbf{F}$, physicists often consider the electric field per unit charge:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q}\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{\|\mathbf{x}\|_2^3}\mathbf{x}$$

Then $\mathbf{E}$ is a vector field on $\mathbb{R}^3$ called the *electric field* of Q.

If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a real-valued function of n variables then its gradient ∇f is a vector field on $\mathbb{R}^n$ and it is called the *gradient vector field* of f.

Definition 5.9. A vector filed $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ is called *conservative* if there exists a real-valued function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{F} = \nabla f^T$. In this situation, f is called a *potential function* for $\mathbf{F}$.

Example 5.9. While not all vector fields are conservative, vector fields arising in physics often are. For example, the gravitational field $\mathbf{F}$ in Example 5.7 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}\nabla f(x, y, z) &= \left(\frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\right) \\ &= \mathbf{F}(x, y, z)^T.\end{aligned}$$

A similar calculation can be done for the electric field $\mathbf{E}$ of a charge Q seen in Example 5.8.

Chapter 6

Multiple Integrals

6.1 Integrability of a bounded function on a closed rectangle

Recall that the definite integral of a non-negative function $f(x) \geqslant 0$ represented the area "under" the curve $y = f(x)$. As we will now see, the double integral of a non-negative real-valued function $f(x, y) \geqslant 0$ represents the volume "under" the surface $z = f(x, y)$.

The goal is to extend the theory of Riemann integrals to real-valued functions in two real variables. Let $a < b$ and $c < d$ be four real numbers and consider the closed rectangle $R = [a, b] \times [c, d]$ and a non-negative continuous function $f \colon R \to \mathbb{R}$. To approximate the volume "under" the surface $z = f(x, y)$, we can use a subdivision of R into smaller rectangles. This is accomplished by dividing the interval $[a, b]$ into n equidistant subintervals and $[c, d]$ into m equidistant subintervals. This will create a grid of rectangles $R_{i,j}$ over the region R. Let $\Delta x = \frac{b-a}{n}$ and $\Delta y = \frac{d-c}{m}$ be the widths of the subintervals along the x-axis and y-axis respectively.

To establish the Riemann sum, we also need to choose a sample point (x_i^*, y_j^*) in each subrectangle $R_{i,j}$. A common choice is the bottom-left corner of each rectangle, but one can choose any point within each rectangle. For each subrectangle $R_{i,j}$, we can now compute the volume of the thin rectangular solid formed by multiplying the function value at the sample point (x_i^*, y_j^*) by the area of the subrectangle $\text{Area}(R_{i,j}) = \Delta x \Delta y$ (see Fig. 6.1). This gives you the approximate volume of the portion of the surface that lies over that rectangular solid.

Summing up all the volumes obtained will then yield an approximation of the volume under the surface $z = f(x, y)$ over the rectangle R.

Definition 6.1 (Double integral)**.** Let $a < b$ and $c < d$ be four real numbers, and $f \colon R = [a, b] \times [c, d] \to \mathbb{R}$ a function. If the limit

$$\iint_R f(x, y)\, dx\, dy = \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*) \Delta x \Delta y$$

exists then we say that $f(x, y)$ is *Riemann integrable* over R. In this case, the number

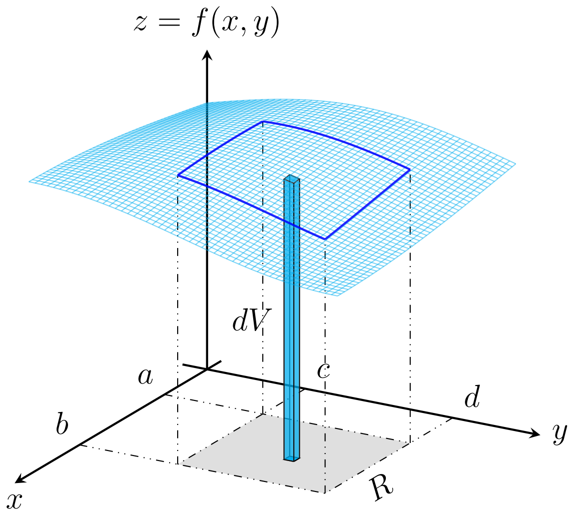


Figure 6.1

$\iint_R f(x, y)\, dx\, dy$ is called the *double integral* of $f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$.

Proposition 6.1. *Let $a < b$ and $c < d$ be four real numbers, $R = [a, b] \times [c, d]$ and $f \colon R \to \mathbb{R}$ be continuous on R. Then, f is integrable on R.*

6.2 Fubini's Theorem for Double Integrals

Let $a < b$ and $c < d$ be four real numbers, and $f \colon R = [a, b] \times [c, d] \to \mathbb{R}$ a continuous function. Then the two functions $g \colon [c, d] \to \mathbb{R}$ and $h \colon [a, b] \to \mathbb{R}$, defined respectively by

$$g(y) = \int_a^b f(x, y) dx \text{ and } h(x) = \int_c^d f(x, y) dy$$

are continuous. Thus, the two numbers

$$\int_c^d g(y) dy = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$$

and

$$\int_a^b h(x) dx = \int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

are well defined. Fubini's theorem says that these two numbers are the same and coincide with the double integral of f over R.

Theorem 6.1 (Fubini's theorem for double integrals – rectangular regions)**.** *Let $a < b$ and $c < d$ be four real numbers, and $f \colon R = [a, b] \times [c, d] \to \mathbb{R}$ a continuous function.*

Then

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)dx\right) dy = \int_a^b \left(\int_c^d f(x,y)dy\right) dx.$$

Example 6.1. Let $R = [0,\pi]\times[0,1]$ and $f\colon R \to \mathbb{R}$ be the continuous function defined by $f(x,y) = x\sin(xy)$. Then, the integral of f over R is given by

$$\iint_R f(x,y)\,dx\,dy = \int_0^\pi \left(\int_0^1 x\sin(xy)\,dy\right)\,dx = \int_0^\pi -\cos(xy)\Big|_{y=0}^{y=1} dx$$

$$= \int_0^\pi (1-\cos x)\,dx = (x-\sin x)\Big|_0^\pi = \pi.$$

We can switch the order of integration because the region R is a rectangle, but in this case, integrating with respect to y first (as we did above) makes the calculation much simpler than starting with x. This shows that sometimes choosing the right order of integration can make a big difference in how easy the problem is to solve.

Example 6.2. Our goal is to find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

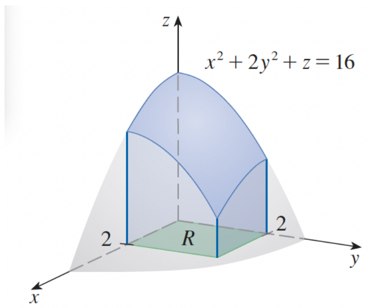


Figure 6.2

We first observe that S is the solid that lies under the surface $z = 16 - x^2 - 2y^2$ and above the square $R = [0,2]\times[0,2]$. (See Figure 6.2.) This solid was considered in Example 1, but we are now in a position to evaluate the double integral using Fubini's Theorem. Therefore

$$\begin{aligned}
V &= \iint_R (16 - x^2 - 2y^2)\,dx\,dy \\
&= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\,dx\,dy \\
&= \int_0^2 \left[16x - \frac{1}{3}x^3 - 2y^2 x\right]_0^2 dy \\
&= \int_0^2 \left(\frac{88}{3} - 4y^2\right) dy \\
&= \left[\frac{88}{3}y - \frac{4}{3}y^3\right]_0^2 \\
&= 48
\end{aligned}$$

Theorem 6.2 (Mean Value Theorem)**.** *Let $a < b$ and $c < d$ be four real numbers and $f : R = [a, b] \times [c, d] \to \mathbb{R}$ a continuous function. Then there exists an element (x_0, y_0) in R such that*

$$\iint_R f(x, y)\,dx\,dy = f(x_0, y_0) \cdot \text{Area}(R)$$

6.3 Double Integrals over general regions

Consider a general region D in $\mathbb{R}^2$ like the one illustrated on the left-hand side of Fig. 6.3. We suppose that D is a bounded region, so D can be enclosed in a rectangular region R as illustrated on the right-hand side of Fig. 6.3. In order to integrate a function $f\colon D \to \mathbb{R}$ over D we define a new function $F\colon R \to \mathbb{R}$ with domain R by

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{if } (x, y) \in R \backslash D. \end{cases} \tag{6.1}$$

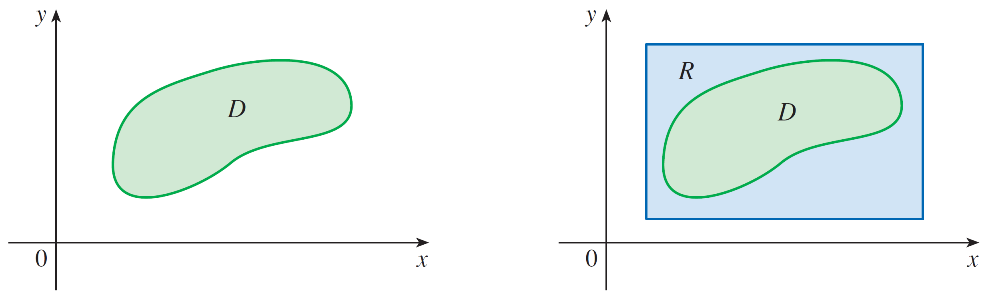


Figure 6.3

Definition 6.2. We say that $f\colon D \to \mathbb{R}$ is *integrable* over the region $D \subseteq \mathbb{R}^2$ if the function F, as defined in (6.1), is integrable over the rectangle R.

Properties of double integrals: Suppose $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are integrable over the region $D \subseteq \mathbb{R}^2$. Then the double integral has the following properties

1. **Linearity**: For all $\alpha, \beta \in \mathbb{R}$ we have
$$\iint_D (\alpha f + \beta g)(x,y)\,dx\,dy = \alpha \iint_D f(x,y)\,dx\,dy + \beta \iint_D g(x,y)\,dx\,dy$$
2. **Monotonicity**: If $f(x,y) \leqslant g(x,y)$ then
$$\iint_D f(x,y)\,dx\,dy \leqslant \iint_D g(x,y)\,dx\,dy.$$
3. **Positivity**: If $f(x,y) \geqslant 0$ then
$$\iint_D f(x,y)\,dx\,dy \geqslant 0.$$
Moreover, if D is open and f is continuous then
$$\iint_D f(x,y)\,dx\,dy = 0 \text{ if and only if } f(x,y) = 0 \text{ for all } (x,y) \in D.$$
4. **Triangle Inequality**: We have
$$\left|\iint_D f(x,y)\,dx\,dy\right| \leqslant \iint_D |f(x,y)|\,dx\,dy$$

Proposition 6.2 (Double Integral over a Subset)**.** *If $f\colon D \to [0,+\infty)$ is bounded and integrable on the bounded subset $D \subseteq \mathbb{R}^2$ and if $f\colon D' \to [0,+\infty)$ is integrable on $D' \subseteq D$, then*
$$\iint_{D'} f(x,y)\,dx\,dy \leqslant \iint_D f(x,y)\,dx\,dy.$$

6.4 Jordan sets

A Jordan set in two dimensions, also known as a Jordan region or Jordan domain, refers to a bounded subset of the plane $\mathbb{R}^2$ that has a well-defined boundary. More formally, a Jordan sets are defined as follows.

Definition 6.3. A bounded subset D of $\mathbb{R}^2$ is a *Jordan set* (in $\mathbb{R}^2$) if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ and closed rectangles $R_1, \ldots, R_k \subseteq \mathbb{R}^2$ such that
$$\partial D \subseteq \bigcup_{j=1}^{k} R_j \quad \text{and} \quad \sum_{j=1}^{k} \operatorname{Area}(R_j) \leqslant \varepsilon.$$

Intuitively, a Jordan set in two dimensions is a well-behaved region with a clear and distinct boundary. This makes it particularly suitable for integration. Examples of Jordan sets include all polygons (triangles, quadrilaterals, etc.), circles, ellipses, and many more.

Theorem 6.3. *Suppose $D \subseteq \mathbb{R}^2$ is a bounded set and $f\colon D \to \mathbb{R}$ is a bounded function. If f is continuous on the interior $\mathring{D}$ and D is a Jordan set then $f(x,y)$ is* *integrable over D.*

In this section, we generalize the definition of the Riemann integral of a bounded function on a closed rectangle to a bounded function on a bounded subset of $\mathbb{R}^2$.

Definition 6.4 (Area)**.** Let D be a Jordan subset of $\mathbb{R}^2$. Then the real number

$$\operatorname{area}(D) = \iint_D 1\, dx\, dy.$$

is called the *area* of D.

Proposition 6.3 (Bounds of the double integral)**.** *Let D be a Jordan subset of $\mathbb{R}^2$, and $f\colon D \to \mathbb{R}$ be a bounded function that is integrable on D. Then,*

$$m \cdot \operatorname{Area}(D) \leqslant \iint_D f(x,y)\, dx\, dy \leqslant M \cdot \operatorname{Area}(D),$$

where $m = \inf\{f(x,y) : (x,y) \in D\}$ and $M = \sup\{f(x,y) : (x,y) \in D\}$.

To find the average value of a function $f(x,y)$ over a region D, divide the double integral $\iint_D f(x,y)\, dx\, dy$ by the area of the region, $\operatorname{Area}(R)$; this yields a single number representing the typical value of the function over D.

Definition 6.5 (Average value)**.** Let D be a Jordan subset of $\mathbb{R}^2$, and $f\colon D \to \mathbb{R}$ be a bounded function that is integrable on D. Then the *average value* of f over D is defined as

$$\text{Average} = \frac{1}{\operatorname{Area}(D)} \iint_D f(x,y)\, dx\, dy$$

Example 6.3. A metal plate in the shape of a rectangle extends from $x = 0$ to $x = 4$ meters and from $y = 0$ to $y = 3$ meters. The temperature at any point on the plate is given by the function

$$T(x,y) = 100 - x^2 - y^2,$$

where $T(x,y)$ is measured in degrees Celsius. Let us find the average temperature of the plate. The average value of $T(x,y)$ over the rectangle $R = [a,b] \times [c,d]$ is given by

$$\text{Average} = \frac{1}{\operatorname{Area}(R)} \iint_R f(x,y)\, dx\, dy.$$

In our case, $\operatorname{Area}(R) = (4-0)(3-0) = 12$. Using Fubini's theorem, we can compute the double integral as

$$\iint_R (100 - x^2 - y^2)\, dx\, dy = \int_0^3 \int_0^4 (100 - x^2 - y^2)\, dx\, dy.$$

The inner integral yields

$$\int_0^4 (100 - x^2 - y^2)\, dx = (100 - y^2)(4) - \frac{64}{3} = 400 - 4y^2 - \frac{64}{3}.$$

Now integrating with respect to y gives

$$\begin{aligned}\int_0^3 \left(400 - 4y^2 - \frac{64}{3}\right) dy &= \int_0^3 \left(\frac{1136}{3} - 4y^2\right) dy \\ &= \left[\frac{1136}{3}y - \frac{4y^3}{3}\right]_0^3 = \frac{1136}{3} \cdot 3 - \frac{4}{3} \cdot 27 = 1100.\end{aligned}$$

So the average temperature equals

$$\frac{1}{12} \cdot 1100 = \frac{1100}{12} = \frac{275}{3} \approx 91.67°\text{C}.$$

6.5 Vertical and horizontal slice methods

Suppose that we have a region D in the xy-plane that is bounded on the left by the vertical line $x = a$, bounded on the right by the vertical line $x = b$ (where $a < b$), bounded below by a curve $y = \varphi_1(x)$, and bounded above by a curve $y = \varphi_2(x)$, as in Fig. 6.4. We will assume that $\varphi_1(x)$ and $\varphi_2(x)$ do not intersect on the open interval (a, b) (they could intersect at the endpoints $x = a$ and $x = b$, though). Then the integral of a continuous function over this region can be computed using the *vertical slice method*.

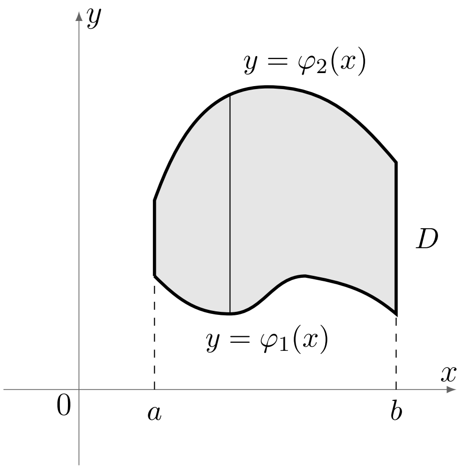


Figure 6.4: Double integral over a non-rectangular region D using the vertical slice method: $\iint_D f(x,y)\,dx\,dy = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx$.

Theorem 6.4 (Vertical Slice Method)**.** *Let a and b be two real numbers, and $\varphi_1, \varphi_2 \colon [a, b] \to \mathbb{R}$ be two continuous functions such that for every $x \in (a, b)$, $\varphi_1(x) < \varphi_2(x)$, and let D be the open bounded subset of $\mathbb{R}^2$ defined by:*

$$D = \left\{(x, y) \in \mathbb{R}^2 : a < x < b, \varphi_1(x) < y < \varphi_2(x)\right\}.$$

Then, for any continuous function

$$f \colon \overline{D} = \left\{(x, y) \in \mathbb{R}^2 \colon a \leqslant x \leqslant b, \varphi_1(x) \leqslant y \leqslant \varphi_2(x)\right\} \to \mathbb{R},$$

we have:

$$\iint_{\overline{D}} f(x,y)\,dx\,dy = \iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right)\,dx.$$

Corollary 6.1. *Let $a < b$ be two real numbers, and let $\varphi_1, \varphi_2 \colon [a,b] \to \mathbb{R}$ be two continuous functions such that for all $x \in (a,b)$, $\varphi_1(x) < \varphi_2(x)$, and let D be the open bounded subset of $\mathbb{R}^2$ defined by*

$$D = \left\{(x,y) \in \mathbb{R}^2 : a < x < b, \varphi_1(x) < y < \varphi_2(x)\right\}.$$

Then,

$$\operatorname{Area}(\overline{D}) = \operatorname{Area}(D) = \int_a^b (\varphi_2(x) - \varphi_1(x))\,dx$$

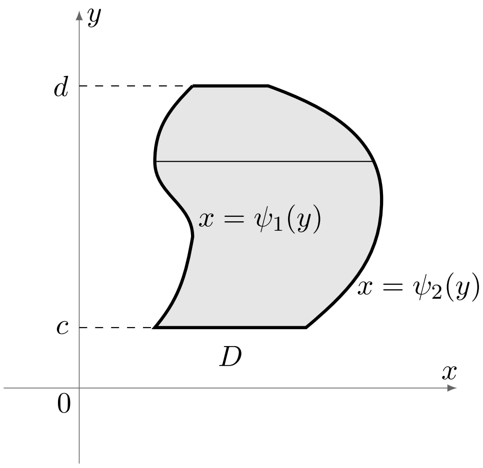


Figure 6.5: Double integral over a non-rectangular region D using the horizontal slice method: $\iint_D f(x,y)\,dx\,dy = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$.

In certain situations, it is advantageous to use horizontal slices instead of vertical ones, see Fig. 6.5.

Theorem 6.5 (Horizontal Slice Method)**.** *Let c and d be two real numbers, and $\psi_1, \psi_2 \colon [c,d] \to \mathbb{R}$ be two continuous functions such that for every $y \in (c,d)$, $\psi_1(y) < \psi_2(y)$, and let D be the open bounded subset of $\mathbb{R}^2$ defined by:*

$$D = \left\{(x,y) \in \mathbb{R}^2 : c < y < d, \psi_1(y) < x < \psi_2(y)\right\}.$$

Then, for any continuous function $f \colon \overline{D} \to \mathbb{R}$ we have:

$$\iint_{\overline{D}} f(x,y)\,dx\,dy = \iint_D f(x,y)\,dx\,dy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right)\,dy.$$

Corollary 6.2. *Let $c < d$ be two real numbers, and let $\psi_1, \psi_2 \colon [c,d] \to \mathbb{R}$ be two continuous functions such that for all $y \in (c,d)$, $\psi_1(y) < \psi_2(y)$, and let D be the open* *bounded subset of* $\mathbb{R}^2$ *defined by*

$$D = \left\{(x, y) \in \mathbb{R}^2 : c < y < d, \psi_1(y) < x < \psi_2(y)\right\}.$$

Then,

$$\text{Area}(\overline{D}) = \text{Area}(D) = \int_c^d (\psi_2(y) - \psi_1(y))\, dy.$$

Example 6.4. Let $D = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1 - x\}$ and $f\colon \overline{D} \to \mathbb{R}$ be the continuous function defined by $f(x, y) = 6^x 2^y$. Hence,

$$\begin{aligned}
\iint_D f(x, y)\mathrm{d}x\ \mathrm{d}y &= \int_0^1 6^x \left(\int_0^{1-x} 2^y\ \mathrm{d}y\right) \mathrm{d}x \\
&= \int_0^1 e^{x \ln 6} \left(\int_0^{1-x} e^{y \ln 2}\ \mathrm{d}y\right) \mathrm{d}x \\
&= \int_0^1 e^{x \ln 6} \frac{1}{\ln 2} \left(e^{(1-x)\ln 2} - 1\right) \mathrm{d}x \\
&= \frac{1}{\ln 2} \int_0^1 \left(e^{x(\ln 2 + \ln 3) - x \ln 2 + \ln 2} - e^{x \ln 6}\right) \mathrm{d}x \\
&= \frac{1}{\ln 2} \int_0^1 \left(2e^{x \ln 3} - e^{x \ln 6}\right) \mathrm{d}x = \frac{1}{\ln 2}\left(\frac{4}{\ln 3} - \frac{5}{\ln 6}\right).
\end{aligned}$$

Example 6.5. Let $D = \{(x, y) \in \mathbb{R}^2 : 0 < x < y < 2x, x^2 + y^2 > 4, xy < 4\}$ (see Fig. 6.6) and $f\colon \overline{D} \to \mathbb{R}$ be the continuous function defined by $f(x, y) = xy$.

Then, denoting by $\varphi_1, \varphi_2\colon [2/\sqrt{5}, 2] \to \mathbb{R}$ the two continuous functions defined respectively by

$$\varphi_1(x) = \begin{cases} \sqrt{4 - x^2} & \text{if } x \in [2/\sqrt{5}, \sqrt{2}] \\ x & \text{if } x \in [\sqrt{2}, 2] \end{cases}$$

and

$$\varphi_2(x) = \begin{cases} 2x & \text{if } x \in [2/\sqrt{5}, \sqrt{2}] \\ 4/x & \text{if } x \in [\sqrt{2}, 2] \end{cases}$$

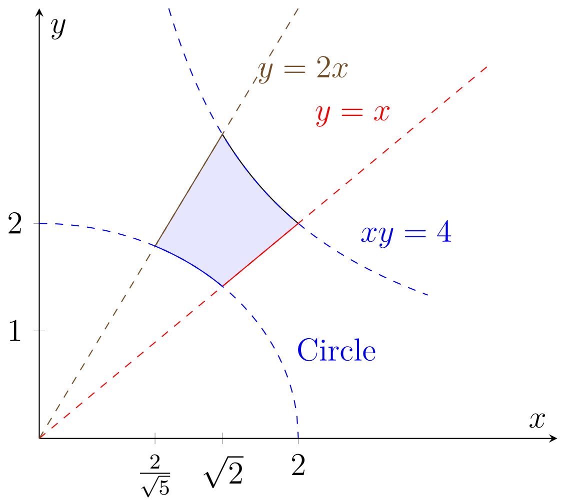


Figure 6.6: Depiction of $D = \{(x, y) : 0 < x < y < 2x, x^2 + y^2 > 4, xy < 4\}$.

we obtain that $D = \left\{(x, y) \in \mathbb{R}^2 \colon 2/\sqrt{5} < x < 2, \varphi_1(x) < y < \varphi_2(x)\right\}$. Consequently,

$$\begin{aligned}
\iint_D f(x, y)\, dx\, dy &= \int_{2/\sqrt{5}}^{2} \left(\int_{\varphi_1(x)}^{\varphi_2(x)} xy\, dy \right) dx \\
&= \int_{2/\sqrt{5}}^{\sqrt{2}} \left(\int_{\sqrt{4-x^2}}^{2x} xy\, dy \right) dx + \int_{\sqrt{2}}^{2} \left(\int_{x}^{4/x} xy\, dy \right) dx \\
&= \frac{1}{2} \int_{2/\sqrt{5}}^{\sqrt{2}} \left(4x^3 - x \cdot (4 - x^2) \right) dx + \frac{1}{2} \int_{\sqrt{2}}^{2} \left(16 - x^3 \right) dx \\
&= \frac{1}{2} \left(\frac{4}{4} x^4 - \frac{1}{2} x^2 \cdot (4 - x^2) \right) \Big|_{2/\sqrt{5}}^{\sqrt{2}} + \frac{1}{2} \left(16 \ln x - \frac{1}{4} x^4 \right) \Big|_{\sqrt{2}}^{2} \\
&= -\frac{3}{5} + 4 \ln 2.
\end{aligned}$$

Example 6.6. Let us find the volume V of the solid bounded by the three coordinate planes and the plane $2x + y + 4z = 4$. The solid is shown in Fig. 6.7 (left) with a

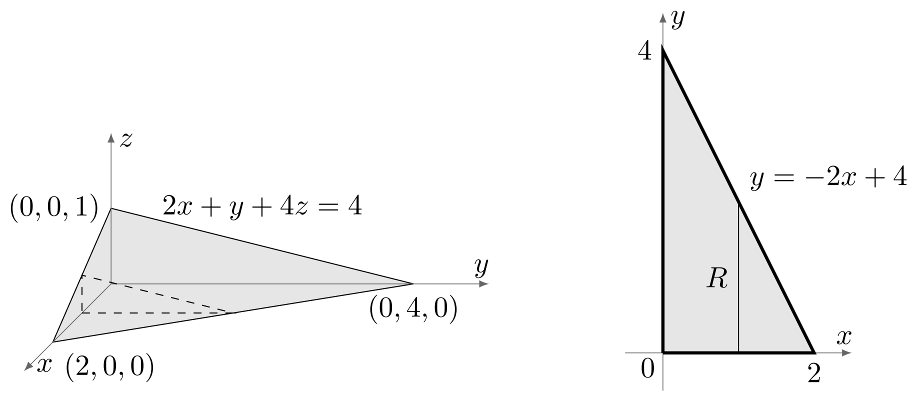


Figure 6.7

typical vertical slice. The volume V is given by the double integral of the function $f(x,y) = z = \frac{1}{4}(4-2x-y)$ over the region $R = \{(x,y) : 0 \leq x \leq 2,\ 0 \leq y \leq -2x+4\}$, shown in Fig. 6.7 (right). Using vertical slices in R gives

$$\begin{aligned} V &= \iint_R \tfrac{1}{4}(4-2x-y)\,dx\,dy \\ &= \int_0^2 \left(\int_0^{-2x+4} \tfrac{1}{4}(4-2x-y)\,dy\right) dx \\ &= \int_0^2 \left(\tfrac{1}{4}(4y-2xy-\tfrac{1}{2}y^2)\Big|_{y=0}^{y=-2x+4}\right) dx \\ &= \int_0^2 (2-2x+\tfrac{1}{2}x^2)\,dx \\ &= 2x - x^2 + \tfrac{1}{6}x^3\Big|_0^2 = \tfrac{4}{3}. \end{aligned}$$

We conclude that the volume V equals $\frac{4}{3}$.

Corollary 6.3 (Fubini's theorem for double integrals – general regions)**.** *Let $a \leqslant b$ and $c \leqslant d$ be real numbers, and $\varphi_1, \varphi_2 \colon (a,b) \to \mathbb{R}$ and $\psi_1, \psi_2 \colon (c,d) \to \mathbb{R}$ be continuous functions such that*

$$\begin{aligned} D = \Big\{(x,y) \in \mathbb{R}^2 : a < x < b, \varphi_1(x) < y < \varphi_2(x)\Big\} \\ = \Big\{(x,y) \in \mathbb{R}^2 : c < y < d, \psi_1(y) < x < \psi_2(y)\Big\}. \end{aligned}$$

Then, for any continuous function $f \colon D \to \mathbb{R}$ we have:

$$\int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right) dx = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right) dy.$$

Example 6.7. We are given the integral

$$I = \int_0^1 \left(\int_{y^3}^1 \frac{36y^8}{1+x^4}\,dx\right) dy.$$

Note that the integrand $\frac{36y^8}{1+x^4}$ separates into a function of y and a function of x, but the limits of the integrals are interdependent. To simplify, we change the order of integration. The region of integration is:

- $y \in [0,1]$
- For each y, $x \in [y^3, 1]$.

This corresponds to:

- $x \in [0,1]$
- For each x, $y \in [0, x^{1/3}]$

So we change the order of integration:

$$I = \int_0^1 \left(\int_0^{x^{1/3}} \frac{36y^8}{1+x^4}\,dy\right) dx.$$

Now factor out the part independent of y:

$$I = \int_0^1 \frac{36}{1+x^4}\left(\int_0^{x^{1/3}} y^8\,dy\right) dx.$$

Evaluate the inner integral:

$$\int_0^{x^{1/3}} y^8\,dy = \left[\frac{y^9}{9}\right]_0^{x^{1/3}} = \frac{x^3}{9}.$$

Substituting:

$$I = \int_0^1 \frac{36}{1+x^4}\cdot\frac{x^3}{9}\,dx = \int_0^1 \frac{4x^3}{1+x^4}\,dx.$$

Let $u = x^4 + 1$, so $du = 4x^3dx$. When $x = 0$, $u = 1$, and when $x = 1$, $u = 2$. Thus,

$$I = \int_1^2 \frac{1}{u}\,du = \ln(2).$$

6.6 Change of variables for Double Integrals

Given the difficulty of evaluating multiple integrals, the reader may be wondering if it is possible to simplify those integrals using a suitable substitution for the variables. The answer is yes, though it is a bit more complicated than the substitution method which you learned in single-variable calculus.

Recall that if you are given, for example, the definite integral

$$\int_1^2 x^3\sqrt{x^2-1}\,dx\ ,$$

then you would make the substitution

$$\begin{aligned} u &= x^2 - 1 \\ du &= 2x\,dx \end{aligned}$$

which changes the limits of integration

$$\begin{aligned} x = 1 &\longrightarrow u = 0 \\ x = 2 &\longrightarrow u = 3 \end{aligned}$$

so that we get

$$\begin{aligned} \int_1^2 x^3\sqrt{x^2-1}\,dx &= \int_1^2 \tfrac{1}{2}x^2\cdot 2x\sqrt{x^2-1}\,dx \\ &= \int_0^3 \tfrac{1}{2}(u+1)\sqrt{u}\,du \\ &= \tfrac{1}{2}\int_0^3 \left(u^{3/2}+u^{1/2}\right)\,du \\ &= \tfrac{14\sqrt{3}}{5}\,. \end{aligned}$$

Let us take a different look at what happened when we did that substitution, which will give some motivation for how substitution works in multiple integrals. First, note that on the interval of integration $[1,2]$, the function $x \mapsto x^2 - 1$ is strictly increasing and maps $[1,2]$ onto $[0,3]$. Hence it has an inverse function $g\colon [0,3] \to [1,2]$, which we can calculate as

$$g(u) \;=\; \sqrt{u+1}\;.$$

Then substituting that expression for x into the function $f(x) = x^3\sqrt{x^2-1}$ gives

$$f(x) \;=\; f(g(u)) \;=\; (u+1)^{3/2}\sqrt{u}\;,$$

and we see that

$$\begin{aligned} dx \;&=\; g\,'(u)\,du \\ dx \;&=\; \tfrac{1}{2}(u+1)^{-1/2}\,du\;, \end{aligned}$$

so since

$$\begin{aligned} 0 &= g^{-1}(1) \\ 3 &= g^{-1}(2) \end{aligned}$$

then performing the substitution as we did earlier gives

$$\begin{aligned} \int_1^2 f(x)\,dx \;&=\; \int_{g^{-1}(1)}^{g^{-1}(2)} f(g(u))\,g\,'(u)\,du \\ &=\; \int_0^3 (u+1)^{3/2}\sqrt{u}\,\cdot\tfrac{1}{2}(u+1)^{-1/2}\,du \\ &=\; \int_0^3 \tfrac{1}{2}(u+1)\sqrt{u}\,du \\ &=\; \tfrac{14\sqrt{3}}{5}\;. \end{aligned}$$

In general, if $g\colon [c,d] \to [a,b]$ is a bijective, differentiable function from an interval $[c,d]$ onto an interval $[a,b]$ with $a = g(c)$ and $b = g(d)$, then

$$\int_a^b f(x)\,dx \;=\; \int_c^d f(g(u))\,g\,'(u)\,du. \tag{6.2}$$

This is called the *change of variable* formula for integrals of single-variable functions. This formula turns out to be a special case of a more general formula which can be used to evaluate multiple integrals. We will state the formulas for double integrals involving real-valued functions of two variables next. We will assume that all the functions involved are continuously differentiable and that the regions and solids involved are Jordan sets (i.e., have "reasonable" boundaries).

Theorem 6.6 (Change of variables for double integrals)**.** *Let D and E be two open Jordan subsets of $\mathbb{R}^2$ and let $\mathbf{\Phi}\colon E \to D$ be a bijection from E to D. Additionally, it is assumed that the vector-valued function $\mathbf{\Phi}$ is of class $C^1(E)$ and its Jacobi determinant* *(cf. Definition 5.5) is bounded over E and satisfies*

$$\det \mathbf{J}_{\mathbf{\Phi}}(u,v) \neq 0$$

for every $(u,v) \in E$. Then, for any continuous and bounded function $f\colon D \to \mathbb{R}$ we have:

$$\iint_D f(x,y)\mathrm{d}x\ \mathrm{d}y = \iint_E f(\mathbf{\Phi}(u,v))\,|\det \mathbf{J}_{\mathbf{\Phi}}(u,v)|\,\mathrm{d}u\ \mathrm{d}v,$$

where the absolute value of the Jacobian appears in the integral.

Example 6.8. Let us evaluate the double integral

$$\iint_D e^{\frac{x-y}{x+y}}\,dx\,dy$$

over the closed region $D = \{(x,y) : x \geq 0, y \geq 0, x+y \leq 1\}$.

First, note that evaluating this double integral without using substitution is probably impossible, at least in a closed form. By looking at the numerator and denominator of the exponent of e, we will try the substitution $u = x - y$ and $v = x + y$. To use the change of variables formula, we need to write both x and y in terms of u and v. So solving for x and y gives $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(v-u)$. This gives the map $\mathbf{\Phi}\colon E \to D$ as

$$\Phi_1(u,v) = \frac{1}{2}(u+v)$$
$$\Phi_2(u,v) = \frac{1}{2}(v-u),$$

where $E = \{(u,v) : 0 \leqslant v \leqslant 1,\ -v \leqslant u \leqslant v\}$. In Fig. 6.8, we see how the mapping $\mathbf{\Phi} = (\Phi_1, \Phi_2)$ maps the region E onto D in a one-to-one manner.

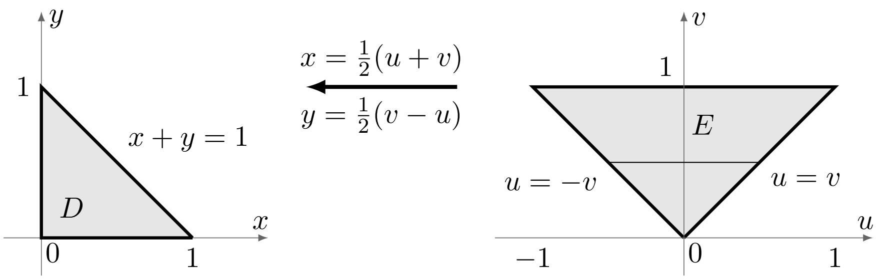


Figure 6.8: The regions D and E

Now we see that

$$\det \mathbf{J}_{\mathbf{\Phi}}(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2},$$

so using the horizontal slices method in E, we have

$$\begin{aligned}
\iint\limits_D e^{\frac{x-y}{x+y}}\, dx\, dy &= \tfrac{1}{2}\iint\limits_E e^{\frac{u}{v}}\, du\, dv \\
&= \tfrac{1}{2}\int_0^1\int_{-v}^{v} e^{\frac{u}{v}}\, du\, dv \\
&= \int_0^1 \left(\tfrac{v}{2} e^{\frac{u}{v}} \Big|_{u=-v}^{u=v} \right) dv \\
&= \int_0^1 \tfrac{v}{2}(e-e^{-1})\, dv \\
&= \frac{v^2}{4}(e-e^{-1})\Big|_0^1 = \frac{1}{4}\left(e-\frac{1}{e}\right) = \frac{e^2-1}{4e}.
\end{aligned}$$

Example 6.9. We aim to evaluate the following double integral:

$$\iint_D e^{x^2+xy+y^2}\, \mathrm{d}x\, \mathrm{d}y$$

where $D = \{(x,y)\in\mathbb{R}^2 : x^2+xy+y^2 < 1, y > 0\}$. Given that for every $(x,y)\in\mathbb{R}^2$:

$$x^2+xy+y^2 = \left(x+\frac{1}{2}y\right)^2 + \frac{3}{4}y^2,$$

we introduce the change of variables $u = x + \frac{1}{2}y$ and $v = \left(\frac{\sqrt{3}}{2}\right)y$, which translates to $x = u - \frac{1}{\sqrt{3}}v$ and $y = \frac{2}{\sqrt{3}}v$. Taking $E = \{(u,v)\in\mathbb{R} : u^2+v^2 < 1, v > 0\}$, we obtain a map $\mathbf{\Phi}\colon E\to D$ defined by

$$\Phi_1(u,v) = u - \frac{1}{\sqrt{3}}v$$

and

$$\Phi_2(u,v) = \frac{2}{\sqrt{3}}v,$$

which is a bijection from E to D. Thus, since for all $(u,v)\in E$,

$$\det \mathbf{J}_{\mathbf{\Phi}}(u,v) = \begin{vmatrix} 1 & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{3}} \end{vmatrix} = \frac{2}{\sqrt{3}} > 0,$$

we obtain, thanks to Theorem 6.6, that

$$\iint_D e^{x^2+xy+y^2}\, \mathrm{d}x\, \mathrm{d}y = \frac{2}{\sqrt{3}}\iint_E e^{u^2+v^2}\, \mathrm{d}u\, \mathrm{d}v.$$

Moreover, as the map

$$\mathbf{\Psi}\colon F = \{(r,\theta)\in\mathbb{R}^2 : 0<r<1, 0<\theta<\pi\}\to E$$

defined by $\Psi_1(r,\theta) = r\cos\theta$ and $\Psi_2(r,\theta) = r\sin\theta$ (polar coordinates) is an injection from F to E and for all $(r, \theta) \in F$:

$$\det \mathbf{J}_{\mathbf{\Psi}}(r, \theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r > 0,$$

we can write, using Theorem 6.5 and Theorem 6.6, that:

$$\iint_E e^{u^2+v^2}\,\mathrm{d}u\,\mathrm{d}v = \iint_F e^{r^2} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi}\left(\int_0^1 re^{r^2}\,\mathrm{d}r\right)\mathrm{d}\theta = \frac{\pi}{2}(e-1).$$

Hence,

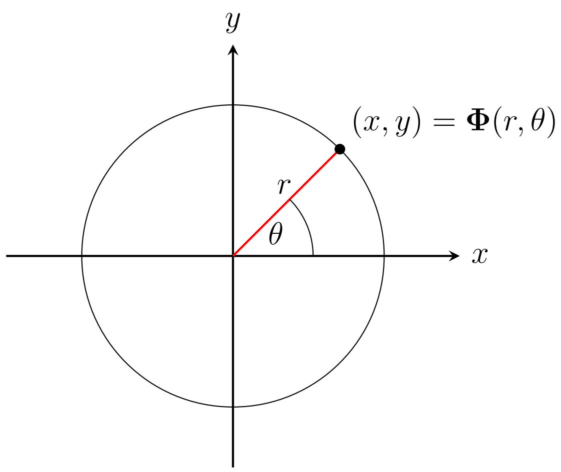
$$\iint_D e^{x^2+xy+y^2}\,\mathrm{d}x\,\mathrm{d}y = \frac{\pi}{\sqrt{3}}(e-1).$$

6.6.1 Polar coordinates

As seen in Example 6.9, the change of variables formula can be used to evaluate double integrals in polar coordinates.

Recall that in polar coordinates, points in the plane $\mathbb{R}^2$ are specified using two parameters:

1. **Radial distance (r):** This measures the distance from the point (x, y) to the origin $(0, 0)$.
2. **Polar angle (θ):** This angle is measured counterclockwise from the positive x-axis to the line segment connecting the origin and the point (x, y).



This representation provides a convenient way to describe circular or radial symmetry and is particularly useful for analyzing problems involving rotation or circular motion.

In mathematical terms, the change from polar coordinates to Cartesian coordinates corresponds to a transformation $\mathbf{\Phi}\colon (0, \infty) \times [0, 2\pi) \to \mathbb{R}^2 \backslash \{(0, 0)\}$, given by

$$\Phi_1(r, \theta) = r\cos\theta \quad \text{and} \quad \Phi_2(r, \theta) = r\sin\theta.$$

Note that the Jacobian determinant of this transformation is

$$\det \mathbf{J}_{\mathbf{\Phi}}(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Following from Theorem 6.6, the change of variables in a double integral from Cartesian to polar coordinates is expressed by the following formula.

Double Integral in Polar Coordinates. We have

$$\iint\limits_D f(x,y)\,dx\,dy \;=\; \iint\limits_E f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta\ , \tag{6.3}$$

where E is a description of the region D in polar coordinates.

Example 6.10. Find the volume V inside the paraboloid $z = x^2 + y^2$ for $0 \le z \le 1$.

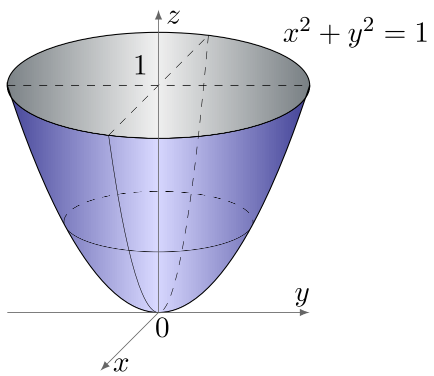


Figure 6.9: The paraboloid $z = x^2 + y^2$.

The volume can be computed using a double integral,

$$V \;=\; \iint\limits_D (1 - z)\,dx\,dy \;=\; \iint\limits_D (1 - (x^2 + y^2))\,dx\,dy,$$

where $D = \{(x,y) : x^2 + y^2 \le 1\}$ is the unit disk in $\mathbb{R}^2$ (see Fig. 6.9). In polar coordinates (r,θ), we know that $x^2 + y^2 = r^2$. So the unit disk D in polar coordiantes corresponds to the set $E = \{(r,\theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$. Thus, changing the double integral into polar coordinates gives

$$\begin{aligned}
V &= \iint\limits_D (1-(x^2+y^2))\,dx\,dy \\
&= \iint\limits_E (1-r^2)\,r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta \\
&= \int_0^{2\pi}\left(\tfrac{r^2}{2}-\tfrac{r^4}{4}\Big|_{r=0}^{r=1}\right)d\theta \\
&= \int_0^{2\pi}\tfrac{1}{4}\,d\theta \\
&= \frac{\pi}{2}.
\end{aligned}$$

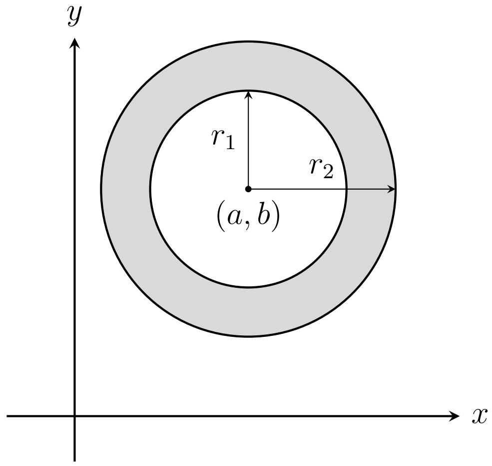
Thus we have $V=\frac{\pi}{2}$.

Let $\mathbf{c}=(a,b)$ be an element of $\mathbb{R}^2$, r a positive real number, and $f:\overline{B(\mathbf{c},r)}\to\mathbb{R}$ a continuous function. Then,

$$\begin{aligned}
&\iint_{B(\mathbf{c},r)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{B(\mathbf{c},r)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{2\pi}\left(\int_0^r f(a+r\cos\theta, b+r\sin\theta)r\,\mathrm{d}r\right)\mathrm{d}\theta \\
&= \int_0^r\left(\int_0^{2\pi} f(a+r\cos\theta, b+r\sin\theta)r\,\mathrm{d}\theta\right)\mathrm{d}r.
\end{aligned}$$

Example 6.11 (Double Integral of a Continuous Function over a Closed Annulus)**.** Let $\mathbf{c}=(a,b)$ be an element of $\mathbb{R}^2$, $r_1<r_2$ two positive real numbers, and let E be the open bounded subset of $\mathbb{R}^2$ defined by

$$E=\left\{(x,y)\in\mathbb{R}^2 : x=a+r\cos\theta, y=b+r\sin\theta, r_1<r<r_2, 0<\theta<2\pi\right\}.$$

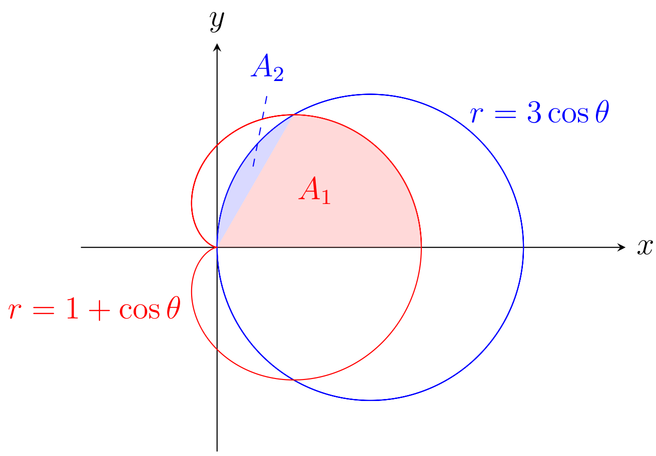


Then, for any continuous function $f : \overline{E} \to \mathbb{R}$, we have:

$$\begin{aligned}
\iint_{\overline{E}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \iint_{E} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{2\pi} \left(\int_{r_1}^{r_2} f(a + r\cos\theta, b + r\sin\theta) r\,\mathrm{d}r \right) \mathrm{d}\theta \\
&= \int_{r_1}^{r_2} \left(\int_0^{2\pi} f(a + r\cos\theta, b + r\sin\theta) r\,\mathrm{d}\theta \right) \mathrm{d}r.
\end{aligned}$$

Example 6.12. Find the area enclosed by the circle $\{(r, \theta) : r = 3\cos\theta,\ -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}\}$ and the cardioid $\{(r, \theta) : r = 1 + \cos\theta,\ 0 \leqslant \theta \leqslant 2\pi\}$.

First, we start by sketching the graphs of the region:



We can see from the symmetry of the graph that we need to find the points of intersection. Setting the two equations equal to each other gives

$$3\cos\theta = 1 + \cos\theta.$$

The two solutions, corresponding to the two points of intersection, are $\theta = \pi/3$ and $\theta = -\pi/3$. The area above the polar axis consists of two parts, with one part defined by the cardioid rom $\theta = 0$ to $\theta = \pi/3$ and the other part defined by the circle from

$\theta = \pi/3$ to $\theta = \pi/2$. By symmetry, the total area is twice the area above the polar axis. Thus, we have

$$A = 2\Bigg(\underbrace{\int_0^{\pi/3}\int_0^{1+\cos\theta} 1r\,dr\,d\theta}_{A_1} + \underbrace{\int_{\pi/3}^{\pi/2}\int_0^{3\cos\theta} 1r\,dr\,d\theta}_{A_2}\Bigg).$$

Evaluating each piece separately, we find that

$$\begin{aligned}
A_1 &= \int_0^{\pi/3}\int_0^{1+\cos\theta} r\,dr\,d\theta \\
&= \int_0^{\pi/3}\left[\frac{1}{2}r^2\right]_0^{1+\cos\theta} d\theta \\
&= \int_0^{\pi/3}\frac{1}{2}(1+\cos\theta)^2\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/3}(1+2\cos\theta+\cos^2\theta)\,d\theta \\
&= \frac{1}{2}\left[\int_0^{\pi/3} 1\,d\theta + 2\int_0^{\pi/3}\cos\theta\,d\theta + \int_0^{\pi/3}\cos^2\theta\,d\theta\right] \\
&= \frac{1}{2}\left[\frac{\pi}{3} + 2\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)\right] = \frac{\pi}{4} + \frac{9\sqrt{3}}{16}.
\end{aligned}$$

A similar calculation reveals that

$$A_2 = \frac{3\pi}{8} - \frac{9\sqrt{3}}{16}.$$

This gives that

$$A = 2A_1 + 2A_2 = \frac{5}{4}\pi.$$

6.7 Triple and Multiple integrals

The purpose of this section is to extend Riemann integrals to real-valued functions in n real variables for arbitrarily large n. We already treated the case $n = 2$ in Section 6.1, where we discusses that the double integral $\iint_R f(x,y)\,dx\,dy$ of a function $f(x,y)$ in two variables over a closed rectangle $R = [a,b]\times[c,d]$ corresponds to the volume "under" the surface $z = f(x,y)$ and "over" the rectangle R.

Now suppose n is an arbitrary positive integer. A subset P of $\mathbb{R}^n$ is called a *closed box* (or sometimes also an *n-dimensional closed hyperrectangle*), if it can be written in the form

$$P = [a_1,b_1]\times\ldots\times[a_n,b_n]$$

where $a_1 < b_1, a_2 < b_2, \ldots, a_n < b_n$ are $2n$ real numbers. Given a real-valued function

$f\colon P \to \mathbb{R}$, the *multiple integral*

$$\int \cdots \int_P f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \tag{6.4}$$

corresponds to the hypervolume "under" the hypersurface $z = f(x_1, x_2, \ldots, x_n)$ and "over" the box P.

To properly define the expression (6.4), we use Riemann sums, just as we did in the $n = 1$ and $n = 2$ cases. We can partition the box P into smaller boxes. This is achieved by dividing each interval $[a_i, b_i]$ into k_i equidistant subintervals, yielding a grid of hyperrectangles $P_{i_1,i_2,\ldots,i_n}$ over the region P. Let $\Delta x_i = \frac{b_i - a_i}{k_i}$ denote the width of the subintervals of the interval $[a_i, b_i]$.

To construct the Riemann sum, we select a sample point $(x^*_{i_1}, x^*_{i_2}, \ldots, x^*_{i_n})$ within each $P_{i_1,i_2,\ldots,i_n}$. A conventional choice is a corner point, but any point within the hyperrectangle suffices. Then for each $P_{i_1,i_2,\ldots,i_n}$, we compute the hypervolume of the thin hyperrectangular solid above it by multiplying the function value at the sample point $(x^*_{i_1}, x^*_{i_2}, \ldots, x^*_{i_n})$ by the hypervolume of the region $P_{i_1,i_2,\ldots,i_n}$, which is $\Delta x_1 \cdot \Delta x_2 \cdot \ldots \cdot \Delta x_n$. Summing all such hypervolumes yields an approximation of the hypervolume under the hypersurface $z = f(x_1, x_2, \ldots, x_n)$ over the box P.

Definition 6.6 (Triple integral)**.** Let $P = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ be a box in $\mathbb{R}^3$ and $f\colon P \to \mathbb{R}$ a function. If the limit

$$\begin{aligned}
&\iiint_P f(x, y, z)\, dx\, dy\, dz \\
&\quad = \lim_{k_1 \to \infty} \sum_{i_1=1}^{k_1} \lim_{k_2 \to \infty} \sum_{i_2=1}^{k_2} \lim_{k_3 \to \infty} \sum_{i_3=1}^{k_3} f(x^*_{i_1}, x^*_{i_2}, x^*_{i_3}) \cdot \Delta x_1 \cdot \Delta x_2 \cdot \Delta x_3
\end{aligned}$$

exists then we say that $f(x, y, z)$ is *Riemann integrable* over P. In this case, the number $\iiint_P f(x, y, z)\, dx\, dy\, dz$ is called the *triple integral* of $f(x, y, z)$ over the box $P = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$.

Definition 6.7 (Multiple integral)**.** Let $P = [a_1, b_1] \times \ldots \times [a_n, b_n]$ be a box in $\mathbb{R}^n$ and $f\colon P \to \mathbb{R}$ a function. If the limit

$$\begin{aligned}
&\int \cdots \int_P f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \\
&\quad = \lim_{k_1 \to \infty} \sum_{i_1=1}^{k_1} \cdots \lim_{k_n \to \infty} \sum_{i_n=1}^{k_n} f(x^*_{i_1}, x^*_{i_2}, \ldots, x^*_{i_n}) \cdot \Delta x_1 \cdot \Delta x_2 \cdot \ldots \cdot \Delta x_n
\end{aligned}$$

exists then we say that $f(x_1, \ldots, x_n)$ is *Riemann integrable* over P. In this case, the number $\int \cdots \int_P f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$ is called the *multiple integral* of $f(x_1, \ldots, x_n)$ over the box $P = [a_1, b_1] \times \ldots \times [a_n, b_n]$.

For continuous functions, we can break down the multi-dimensional integral into a sequence of one-dimensional integrals, and one can do so in any order. This makes calculating complicated integrals in multiple dimensions much more manageable by breaking them down into simpler steps.

Theorem 6.7 (Fubini's theorem for triple integrals)**.** *Let $P = [a, b] \times [c, d] \times [e, f]$ be a closed box in $\mathbb{R}^3$ and $f\colon P \to \mathbb{R}$ a continuous function. Then f is Riemann integrable over P and*

$$\begin{aligned}
\iiint_P f(x,y,z)\,dx\,dy\,dz &= \int_e^f \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx \right) dy \right) dz \\
&= \int_c^d \left(\int_e^f \left(\int_a^b f(x,y,z)\,dx \right) dz \right) dy \\
&= \int_c^d \left(\int_a^b \left(\int_e^f f(x,y,z)\,dz \right) dx \right) dy \\
&= \int_e^f \left(\int_a^b \left(\int_c^d f(x,y,z)\,dy \right) dx \right) dz \\
&= \int_a^b \left(\int_e^f \left(\int_c^d f(x,y,z)\,dy \right) dz \right) dx \\
&= \int_a^b \left(\int_c^d \left(\int_e^f f(x,y,z)\,dz \right) dy \right) dx.
\end{aligned}$$

Example 6.13. Let us evaluate the triple integral

$$\iiint_R (xy + z)\,dx\,dy\,dz$$

over the box $R = [0, 1] \times [0, 2] \times [0, 3]$. We have

$$\begin{aligned}
\iiint_R (xy + z)\,dx\,dy\,dz &= \int_0^3 \int_0^2 \int_0^1 (xy + z)\,dx\,dy\,dz \\
&= \int_0^3 \int_0^2 \left(\tfrac{1}{2}x^2 y + xz \Big|_{x=0}^{x=1} \right) dy\,dz \\
&= \int_0^3 \int_0^2 \left(\tfrac{1}{2}y + z \right) dy\,dz \\
&= \int_0^3 \left(\tfrac{1}{4}y^2 + yz \Big|_{y=0}^{y=2} \right) dz \\
&= \int_0^3 (1 + 2z)\,dz \\
&= z + z^2 \Big|_0^3 = 12.
\end{aligned}$$

Theorem 6.8 (Fubini's theorem for multiple integrals)**.** *Let $P = [a_1, b_1] \times \ldots \times [a_n, b_n]$ be a closed box in $\mathbb{R}^n$ and $f\colon P \to \mathbb{R}$ a continuous function. Then f is Riemann integrable over P and*

$$\begin{aligned}
&\int \cdots \int_P f(x_1, \ldots, x_n)\,dx_1 \cdots dx_n \\
&\qquad = \int_{a_n}^{b_n} \left(\int_{a_{n-1}}^{b_{n-1}} \left(\ldots \left(\int_{a_1}^{b_1} f(x_1, \ldots, x_n)\,dx_1 \right) \ldots \right) dx_{n-1} \right) dx_n.
\end{aligned}$$

Moreover, the order of integration does not affect the value of the multiple integral, and one can rearrange the individual integrals in any order.

6.8 Multiple Integrals over general regions

Let S be a bounded subset of $\mathbb{R}^n$, where n is any positive integer. Since S is a bounded region, it can be enclosed in a hyperrectangle $R = [a_1, b_1] \times \ldots \times [a_n, b_n]$ in $\mathbb{R}^n$. To integrate a function $f\colon S \to \mathbb{R}$ over the region S, we define a new function $F\colon R \to \mathbb{R}$ with domain R by

$$F(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in S, \\ 0 & \text{if } \mathbf{x} \in R \backslash S, \end{cases} \tag{6.5}$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)$.

Definition 6.8. We say that $f\colon S \to \mathbb{R}$ is *integrable* over the region $S \subseteq \mathbb{R}^n$ if the function F, as defined in (6.5), is integrable over the hyperrectangle R.

Properties of multiple integrals: Suppose $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ are integrable over the region $S \subseteq \mathbb{R}^n$. Then the multiple integral has the following properties:

1. **Linearity**: For all $\alpha, \beta \in \mathbb{R}$ we have

$$\int \cdots \int_S (\alpha f + \beta g)\, dx_1 \cdots dx_n = \alpha \int \cdots \int_S f\, dx_1 \cdots dx_n + \beta \int \cdots \int_S g\, dx_1 \cdots dx_n.$$

2. **Monotonicity**: If $f(\mathbf{x}) \leqslant g(\mathbf{x})$ for all $\mathbf{x} \in S$ then

$$\int \cdots \int_S f\, dx_1 \cdots dx_n \leqslant \int \cdots \int_S g\, dx_1 \cdots dx_n.$$

3. **Positivity**: If $f(\mathbf{x}) \geqslant 0$ then

$$\int \cdots \int_S f\, dx_1 \cdots dx_n \geqslant 0.$$

Moreover, if S is open and f is continuous then

$$\int \cdots \int_S f\, dx_1 \cdots dx_n = 0 \;\text{ if and only if } f(\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in S.$$

4. **Triangle inequality**:

$$\left| \int \cdots \int_S f\, dx_1 \cdots dx_n \right| \leqslant \int \cdots \int_S |f|\, dx_1 \cdots dx_n.$$

6.9 Computing Triple Integrals

A more complicated case of a triple integral is where S is a solid which is bounded below by a surface $z = g_1(x, y)$, bounded above by a surface $z = g_2(x, y)$, y is bounded between two curves $h_1(x)$ and $h_2(x)$, and x varies between a and b. Then

$$\iiint\limits_S f(x,y,z)\,dx\,dy\,dz \;=\; \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\,dy\,dx\ . \tag{6.6}$$

Notice in this case that the first iterated integral will result in a function of x and y (since its limits of integration are functions of x and y), which then leaves you with a double integral of a type that we learned how to evaluate in Section 6.5. There are, of course, many variations on this case (for example, changing the roles of the variables x, y, z), so as you can probably tell, triple integrals can be quite tricky.

Example 6.14. Let S denote the solid in the first octant underneath the plane $x+y+z=1$, see Fig. 6.10. Let us evaluate the triple integral

$$\iiint_S (x+y+z)\,dz\,dy\,dx$$

over S.

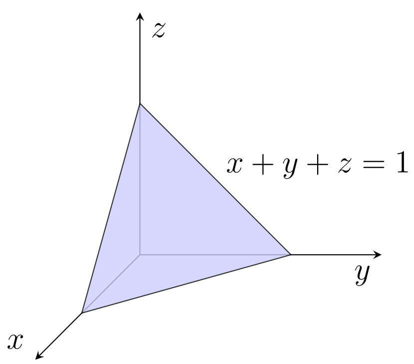


Figure 6.10: A depiction of the solid in the first octant underneath the plane $x+y+z=1$.

First, we observe that $S = \{(x,y,z) : 0 < x < 1,\ 0 < y < 1-x,\ 0 < z < 1-x-y\}$.

Thus we have

$$\begin{aligned}
\iiint_S (x+y+z)\,dz\,dy\,dx &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)\,dz\,dy\,dx \\
&= \int_0^1 \int_0^{1-x} \left((x+y)z + \tfrac{1}{2}z^2 \Big|_{z=0}^{z=1-x-y} \right) dy\,dx \\
&= \int_0^1 \int_0^{1-x} \left((x+y)(1-x-y) + \tfrac{1}{2}(1-x-y)^2 \right) dy\,dx \\
&= \int_0^1 \int_0^{1-x} \left(\tfrac{1}{2} - xy - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 \right) dy\,dx \\
&= \int_0^1 \left(\tfrac{1}{2}y - \tfrac{1}{2}xy^2 - \tfrac{1}{2}x^2y - \tfrac{1}{6}y^3 \Big|_{y=0}^{y=1-x} \right) dx \\
&= \int_0^1 \left(\tfrac{1}{6}x^3 - \tfrac{1}{2}x + \tfrac{1}{3} \right) dx \\
&= \tfrac{1}{24}x^4 - \tfrac{1}{4}x^2 + \tfrac{1}{3}x \Big|_0^1 \;=\; \tfrac{1}{8}.
\end{aligned}$$

Note that the volume V of a solid in $\mathbb{R}^3$ is given by

$$V = \iiint_S 1\,dx\,dy\,dz. \tag{6.7}$$

Since the function being integrated is the constant 1, the above triple integral reduces to a double integral of the types that we considered in the previous section if the solid is bounded below and above by surfaces $z = g_1(x,y)$ and $z = g_2(x,y)$, respectively, with y bounded between two curves $h_1(x)$ and $h_2(x)$, and x varies between a and b. Then

$$\begin{aligned}
V = \iiint_S 1\,dx\,dy\,dz &= \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} 1\,dz\,dy\,dx \\
&= \int_a^b \int_{h_1(x)}^{h_2(x)} \left(g_2(x,y) - g_1(x,y) \right) dy\,dx.
\end{aligned}$$

Example 6.15. Find the volume of the solid S bounded by the three coordinate planes, bounded above by the plane $x+y+z=2$, and bounded below by the plane $x+y-z=0$.

The volume is trapped between the upper plane $z = 2-x-y$ and the lower plane $z = x+y$. Thus the limits for z are

$$x+y \leqslant z \leqslant 2-x-y. \tag{6.8}$$

Note that (6.8) can only hold if $x+y \leqslant 2-x-y$, which is equivalent to $y \leqslant 1-x$. Together with the assumption $y \geqslant 0$, we obtain the limits for y,

$$0 \leqslant y \leqslant 1-x. \tag{6.9}$$

Finally, (6.9) implies $x \leqslant 1$, which together with $x \geqslant 0$ gives the limits for x,

$$0 \leqslant x \leqslant 1. \tag{6.10}$$

Combining (6.8), (6.9), and (6.10), we can describe the solid S as

$$S = \{(x, y, z) : 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ x + y \leqslant z \leqslant 2 - x - y\}.$$

Hence

$$\begin{aligned}
\text{Volume}(S) &= \iiint_S 1\, dx\, dy\, dz \\
&= \int_0^1 \int_0^{1-x} \int_{x+y}^{2-x-y} 1\, dz\, dy\, dx \\
&= \int_0^1 \int_0^{1-x} (2 - 2x - 2y)\, dy\, dx \\
&= \int_0^1 \left(2y - 2xy - y^2 \Big|_{y=0}^{y=1-x} \right) dx \\
&= \int_0^1 \left(2(1-x) - 2x(1-x) - (1-x)^2 \right) dx \\
&= \int_0^1 (x-1)^2\, dx \\
&= \frac{1}{3}.
\end{aligned}$$

6.10 Change of Variables for Triple Integrals

Theorem 6.9. *Suppose $\mathbf{\Phi}\colon E \to D$ is a bijection from an open bounded region E to an open bounded region D. Additionally, it is assumed that $\mathbf{\Phi}$ is of class $C^1(E)$ and its Jacobi determinant is bounded over E and satisfies*

$$\det \mathbf{J}_{\mathbf{\Phi}}(u, v, w) \neq 0$$

for every $(u, v, w) \in E$. Then, for any continuous and bounded function $f\colon D \to \mathbb{R}$ we have:

$$\iiint_D f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iint_E f(\mathbf{\Phi}(u, v, w))\, |\det \mathbf{J}_{\mathbf{\Phi}}(u, v, w)|\, \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}w,$$

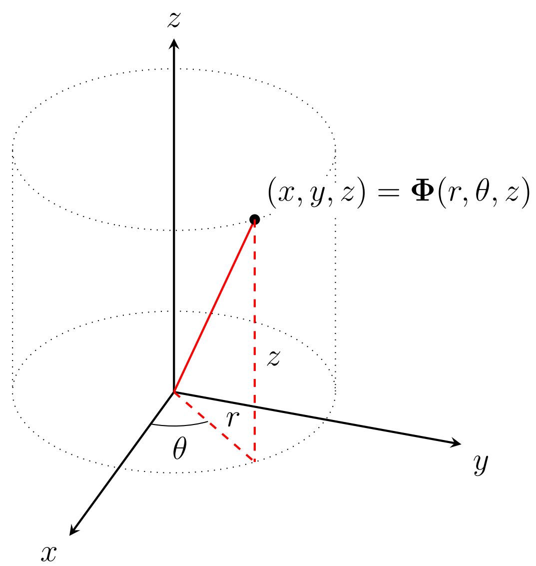
where the absolute value of the Jacobian appears in the integral.

Two examples of change of variables for triple integrals are transitioning from Cartesian coordinates to cylindrical coordinates and from Cartesian coordinates to spherical coordinates.

6.10.1 Cylindrical Coordinates

Cylindrical coordinates are a three-dimensional coordinate system where a point (x, y, z) in $\mathbb{R}^3$ space is described by three quantities:

1. **Radial distance from the z-axis (r):** This represents the shortest distance from the point to the z-axis. It is a non-negative real number.
2. **Polar angle (θ):** Also called the azimuthal angle, this angle is measured in the xy-plane counterclockwise from the positive x-axis to the projection of the point onto the xy-plane. It ranges from 0 to 2π.
3. **Height (z):** The third coordinate remains the same as in Cartesian coordinates and measures the height above or below the xy-plane.



The conversion from Cartesian coordinates (x, y, z) to cylindrical coordinates (r, θ, z) is given by a transformation $\mathbf{\Phi}\colon (0, \infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3 \backslash \{(0, 0, 0)\}$ defined as

$$\begin{aligned}
\Phi_1(r, \theta, z) &= r\cos(\theta) \\
\Phi_2(r, \theta, z) &= r\sin(\theta) \\
\Phi_3(r, \theta, z) &= z
\end{aligned}$$

Cylindrical coordinates are particularly useful for describing objects with cylindrical symmetry, and for simplifying triple integrals involving rotational symmetry or cylindrical shapes.

The Jacobian determinant of $\mathbf{\Phi}$ is

$$\det \mathbf{J}_{\mathbf{\Phi}}(r, \theta, z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Thus, in light of Theorem 6.9, the change of variables in a triple integral from Cartesian to cylindrical coordinates is as follows.

Triple Integral in Cylindrical Coordinates. We have

$$\iiint\limits_D f(x, y, z)\, dx\, dy\, dz \;=\; \iiint\limits_E f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta, dz \;,$$

where E is a description of the region D in cylindrical coordinates.

In particular, if $R \in (0, +\infty)$, $h_1 < h_2$ are two real numbers, and D is the cylinder defined by

$$D = \{(x, y) \in \mathbb{R}^3 : x^2 + y^2 < R^2, h_1 < z < h_2\},$$

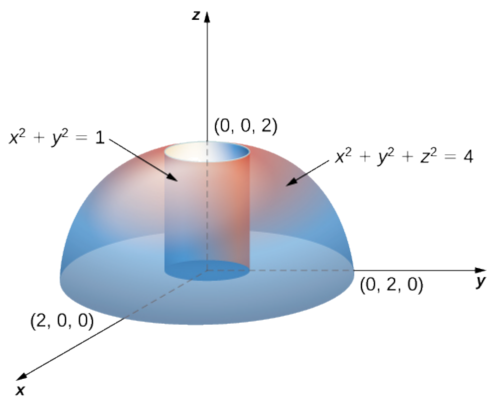
then, for any continuous and bounded function $f\colon D \to \mathbb{R}$, we have:

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \int_0^{2\pi} \left(\int_{h_1}^{h_2} \left(\int_0^R f(r\cos\theta, r\sin\theta, z) r\, dr \right) dz \right) d\theta$$

This equality remains valid even if the order of integration is permuted. In particular,

$$\text{Volume}(D) = \iiint_D dx\, dy\, dz = \int_0^{2\pi} \left(\int_{h_1}^{h_2} \left(\int_0^R r\, dr \right) dz \right) d\theta = \pi R^2 (h_2 - h_1).$$

Example 6.16. Let D be the solid bounded below by the xy-plane, above by the sphere $x^2 + y^2 + z^2 = 4$ and on the sides by the cylinder $x^2 + y^2 = 1$ (see Example 6.16).



Let us use cylindrical coordinates to tackle this problem. Note that the equation for the sphere is

$$x^2 + y^2 + z^2 = 4 \,\text{or}\, r^2 + z^2 = 4,$$

and the equation for the cylinder is

$$x^2 + y^2 = 1 \,\text{or}\, r^2 = 1.$$

Thus, the solid D expressed in cylindrical coordinates corresponds to the set

$$E = \{(r, \theta, z) | 0 \leqslant z \leqslant \sqrt{4 - r^2},\ 0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi\}.$$

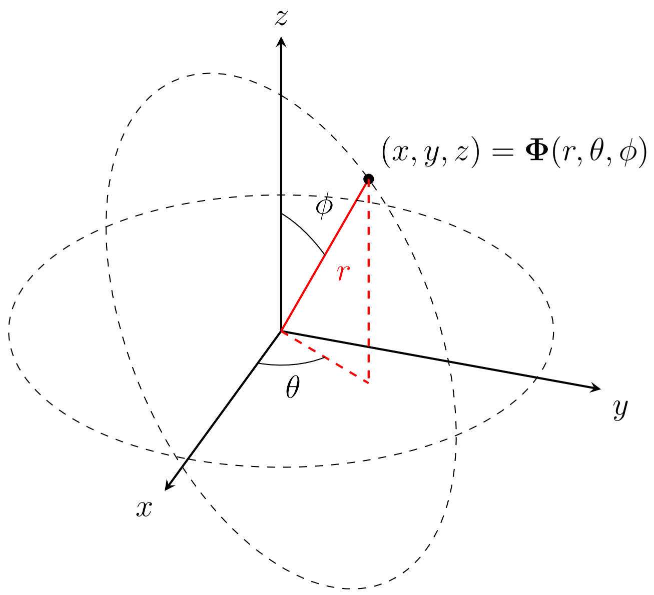
Hence the integral for the volume is

$$\begin{aligned}
\text{Volume}(D) &= \iiint_D 1\,dx\,dy\,dz \\
&= \iiint_E r\,dr\,dz\,d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left(rz \Big|_{z=0}^{z=\sqrt{4-r^2}} \right) dr\,d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left(r\sqrt{4-r^2} \right) dr\,d\theta \\
&= \int_0^{2\pi} \left(\frac{8}{3} - \sqrt{3} \right) d\theta \\
&= 2\pi \left(\frac{8}{3} - \sqrt{3} \right).
\end{aligned}$$

6.10.2 Spherical Coordinates

Spherical coordinates are a three-dimensional coordinate system where a point in space is described using the following three quantities:

1. **Radial distance from the origin (r):** This represents the distance from the origin to the point. It is a non-negative real number.
2. **Polar angle (θ):** Also known as the azimuthal angle, this angle is measured in the xy-plane counterclockwise from the positive x-axis to the projection of the point. It ranges from 0 to 2π.
3. **Elevation Angle (ϕ):** Also known as the zenith angle, this angle is measured from the positive z-axis to the point. It ranges from 0 (the positive z-axis) to π (the negative z-axis).



The conversion from spherical coordinates (r, θ, ϕ) to Cartesian coordinates (x, y, z) is given by the transformation $\mathbf{\Phi}\colon (0, \infty) \times [0, 2\pi) \times (0, \pi) \to \mathbb{R}^3 \backslash \{(0, 0, z) : z \in \mathbb{R}\}$ defined as

$$\begin{aligned} \Phi_1(r, \theta, \phi) &= r \sin(\phi) \cos(\theta) \\ \Phi_2(r, \theta, \phi) &= r \sin(\phi) \sin(\theta) \\ \Phi_3(r, \theta, \phi) &= r \cos(\phi). \end{aligned}$$

Spherical coordinates are particularly useful for describing objects with spherical symmetry, such as spheres, and for simplifying calculations involving spherical shapes or symmetrical distributions of points in space.

The Jacobian determinant of $\mathbf{\Phi}$ is

$$\det \mathbf{J}_{\mathbf{\Phi}}(r, \theta, \phi) = \begin{vmatrix} \sin(\phi) \cos(\theta) & -r \sin(\phi) \sin(\theta) & -r \cos(\phi) \cos(\theta) \\ \sin(\phi) \sin(\theta) & r \sin(\phi) \cos(\theta) & -r \cos(\phi) \sin(\theta) \\ \cos(\phi) & 0 & r \sin(\phi) \end{vmatrix} = r^2 \sin \phi.$$

By Theorem 6.9, the change of variables in a triple integral from Cartesian to spherical coordinates is as follows.

Triple Integral in Spherical Coordinates. We have

$$\iiint\limits_D f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint\limits_E f(r \sin(\phi) \cos(\theta), r \sin(\phi) \sin(\theta), r \cos(\phi))\, r^2 \sin(\phi)\, dr\, d\theta, d\phi\ ,$$

where E is a description of the region D in spherical coordinates.

Using spherical coordinates, we can easily compute the volume of a ball in $\mathbb{R}^3$, i.e.,

$$\begin{aligned}\text{Volume}(B(\mathbf{0}, r_0)) &= \iiint_{B(0,r_0)} dx\, dy\, dz \\ &= \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^{r_0} r^2 \sin\phi\, dr \right) d\phi \right) d\theta \\ &= \frac{4\pi r_0^3}{3}.\end{aligned}$$

Example 6.17. Let us evaluate the triple integral $\iiint_E 16z\, dx\, dy\, dz$ where E is the upper half of the sphere $x^2 + y^2 + z^2 = 1$. Since we are integrating over the upper half of a sphere, it is beneficial to use spherical coordinates to evaluate this integral. The description of E in shperical coordinates is

$$E = \{(r, \theta, \phi) : 0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \frac{\pi}{2}\}.$$

The integral is then:

$$\begin{aligned}\iiint_E 16z\, dx\, dy\, dz &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 r^2 \sin\phi \cdot (16r\cos\phi)\, dr\, d\theta\, d\phi \\ &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 8r^3 \sin(2\phi)\, dr\, d\theta\, d\phi \\ &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 2\sin(2\phi)\, d\theta\, d\phi \\ &= \int_0^{\frac{\pi}{2}} 4\pi \sin(2\phi)\, d\phi \\ &= -2\pi\cos(2\phi)\Big|_0^{\frac{\pi}{2}} \\ &= 4\pi\end{aligned}$$

Example 6.18. Let us find $\iiint_D zx\, dx\, dy\, dz$ where D is inside both $x^2 + y^2 + z^2 = 4$ and the cone (pointing upward) that makes an angle of $\frac{\pi}{3}$ with the negative z-axis and has $x \leqslant 0$.

First, we need to determine the limits of integration. The region D is basically an upside-down ice cream cone that has been cut in half so that only the portion with $x \leqslant 0$ remains. Therefore, because we are inside a portion of a sphere of radius 2 we must have:

$$0 \leqslant r \leqslant 2$$

For ϕ we need to be careful. The problem statement says that the cone makes an angle of $\frac{\pi}{3}$ with the negative z-axis. However, remember that ϕ is measured from the positive z-axis. Therefore, the first angle, as measured from the positive z-axis, that will "start" the cone will be $\phi = \frac{2\pi}{3}$ and it goes to the negative z-axis. Therefore, we get the following limits for ϕ:

$$\frac{2\pi}{3} \leqslant \phi \leqslant \pi$$

Finally, for θ we can use the fact that we are also told that $x \leqslant 0$. This means we are to the left of the y-axis and so the range of θ must be:

$$\frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}$$

In summary, we have found a description of the set D in polar coordinates, given by

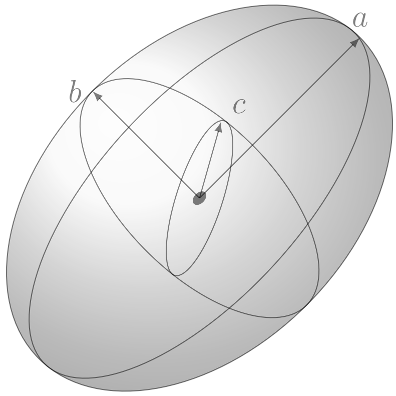
$$E = \{(r, \theta, \phi) : 0 \leqslant r \leqslant 2,\ \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2},\ \frac{2\pi}{3} \leqslant \phi \leqslant \pi\}.$$

Now that we have the limits we can evaluate the integral in spherical coordinates:

$$\begin{aligned}
\iiint_D z\, x\, dx\, dy\, dz &= \iiint_E (r\cos\phi)\,(r\sin\phi\cos\theta)\, r^2 \sin\phi\, dr\, d\theta\, d\phi \\
&= \int_{\frac{2\pi}{3}}^{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^2 (r\cos\phi)\,(r\sin\phi\cos\theta)\, r^2 \sin\phi\, dr\, d\theta\, d\phi \\
&= \int_{\frac{2\pi}{3}}^{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^2 r^4 \cos\phi \sin^2\phi \cos\theta\, dr\, d\theta\, d\phi \\
&= \int_{\frac{2\pi}{3}}^{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{32}{5} \cos\phi \sin^2\phi \cos\theta\, d\theta\, d\phi \\
&= \int_{\frac{2\pi}{3}}^{\pi} -\frac{64}{5} \cos\phi \sin^2\phi\, d\phi \\
&= -\frac{64}{15} \sin^3\phi \Big|_{\frac{2\pi}{3}}^{\pi} \\
&= \frac{8\sqrt{3}}{5}.
\end{aligned}$$

Example 6.19 (Volume of an Ellipsoid)**.** Let a, b, c be three positive real numbers, and D be the ellipsoid defined by

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1 \right\}.$$



We can calculate the volume of D using a triple integral. We have

$$\begin{aligned}\text{Volume}(D) &= \iiint_D 1 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_{-a}^{a}\left(\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}}\left(\int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} 1\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{-a}^{a}\left(\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} 2c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{-a}^{a}\left(\int_{-\pi/2}^{\pi/2} 2bc\left(1-\frac{x^2}{a^2}\right)\cos^2\theta\,\mathrm{d}\theta\right)\mathrm{d}x,\end{aligned}$$

where in the last step we have used the substitution $y = b\sqrt{1-\frac{x^2}{a^2}}\sin\theta$. We can now separate the two integrals and get

$$\begin{aligned}\int_{-a}^{a}\left(\int_{-\pi/2}^{\pi/2} 2bc\left(1-\frac{x^2}{a^2}\right)\cos^2\theta\,\mathrm{d}\theta\right)\mathrm{d}x &= 2bc\left(\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)\mathrm{d}x\right)\cdot\left(\int_{-\pi/2}^{\pi/2}\cos^2\theta\,\mathrm{d}\theta\right) \\ &= \frac{4\pi abc}{3}.\end{aligned}$$

Example 6.20 (Volume of the Sphere in $\mathbb{R}^n$)**.** Let r be a positive real number and denote by $V_n(r)$ the volume of the sphere $B_n(\mathbf{0}, r) = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sqrt{x_1^2 + \ldots + x_n^2} < r\right\}$. Since $V_2(r) = \pi r^2$ and $V_3(r) = \frac{4\pi}{3}r^3$, it appears that $V_n(r) = \alpha_n r^n$ with $\alpha_n \in (0, +\infty)$.

Indeed, assume the result is true for an integer $n \geqslant 2$. Then,

$$\begin{aligned}
V_{n+1}(r) &= \int_{B_{n+1}(\mathbf{0},r)} \cdots \int_{-r} dx_1 \ldots dx_{n+1} \\
&= \int^{r}_{B_n\left(0,\sqrt{r^2-x_{n+1}^2}\right)} \left(\int_1 \ldots dx_n\right) dx_{n+1} \\
&= \int_{-r}^{r} \alpha_n \left(r^2 - x_{n+1}^2\right)^{n/2} dx_{n+1} \\
&= \int_{-\pi/2}^{\pi/2} \alpha_n r^{n+1} \cos^{n+1}\theta \, d\theta \quad (\text{with } x_{n+1} = r\sin\theta) \\
&= \left(\int_{-\pi/2}^{\pi/2} \alpha_n \cos^{n+1}\theta \, d\theta\right) \cdot r^{n+1} = \alpha_{n+1} r^{n+1}.
\end{aligned}$$

Since the result is true for $n = 2$, we have demonstrated by induction that for every integer $n \geqslant 2$, a positive real number α_n can be associated such that $V_n(r) = \alpha_n r^n$. Moreover, for every $n \geqslant 3$, we have:

$$\alpha_n = \alpha_{n-1} \cdot \int_{-\pi/2}^{\pi/2} \cos^n\theta \, d\theta.$$

We now propose to calculate the α_n. For this purpose, consider the sequence (β_n) defined by

$$\beta_n = \int_{-\pi/2}^{\pi/2} \cos^n\theta \, d\theta = 2\int_0^{\pi/2} \cos^n\theta \, d\theta.$$

It is known from Exercise 7.5.4 that for every integer $n \geqslant 2$:

$$\beta_n = \frac{n-1}{n}\beta_{n-2};$$

which implies, since $\beta_0 = \pi$ and $\beta_1 = 2$, that for every $n \in \mathbb{N}^*$,

$$\beta_n\beta_{n-1} = \frac{2\pi}{n}.$$

From this result, we deduce that for every integer $n \geqslant 4$:

$$\alpha_n = \frac{2\pi}{n}\alpha_{n-2}.$$

Finally, since $\alpha_2 = \pi$ and $\alpha_3 = \frac{4\pi}{3}$, we find that for every integer $n \geqslant 2$:

$$\alpha_n = \begin{cases} \frac{2\pi}{n} \cdot \frac{2\pi}{n-2} \cdot \ldots \cdot \frac{2\pi}{2} & \text{if } n \text{ is even} \\ \frac{2\pi}{n} \cdot \frac{2\pi}{n-2} \cdot \ldots \cdot \frac{2\pi}{3} \cdot 2 & \text{if } n \text{ is odd.} \end{cases}$$

6.11 Center of Mass in $\mathbb{R}^2$

Suppose R is a region in $\mathbb{R}^2$ hat represents a thin, flat plate (or *lamina*), and assume the density of this plate is a continuous function $\delta = \delta(x, y)$ of the coordinates (x, y)

of points in R. The the *mass* of plate R is the double integral of the density function $\delta(x,y)$ over R, i.e.,

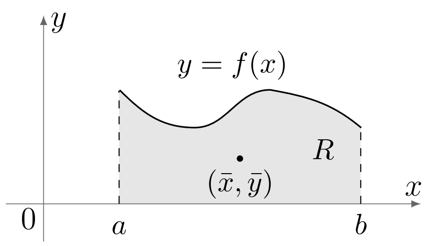
$$M = \text{Mass}(R) = \iint\limits_R \delta(x,y)\,dx\,dy.$$

The coordinates $(\bar{x},\bar{y})$ of the *center of mass* of R are given by

$$\bar{x} = \frac{M_y}{M} \quad \text{and} \quad \bar{y} = \frac{M_x}{M}\ , \tag{6.11}$$

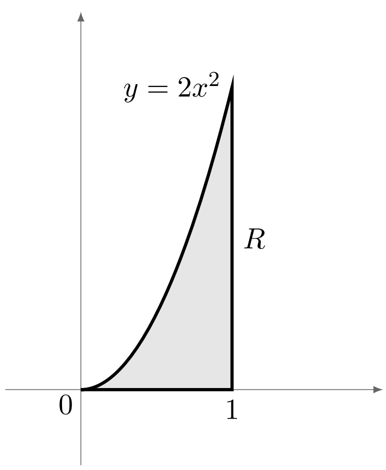
where

$$M_y = \iint\limits_R x\delta(x,y)\,dx\,dy\ , \qquad \text{and} \qquad M_x = \iint\limits_R y\delta(x,y)\,dx\,dy. \tag{6.12}$$



The quantities M_x and M_y are called the *moments* (or *first moments*) of the region R about the x-axis and y-axis, respectively. In the special case where the density function $\delta(x,y)$ is a constant function on the region R, the center of mass $(\bar{x},\bar{y})$ is called the *centroid* of R.

Example 6.21. Find the center of mass of the region $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2x^2\}$, if the density function at (x,y) is $\delta(x,y) = x + y$.



The region R is shown above. The mass of R is

$$\begin{aligned} M &= \iint\limits_R \delta(x,y)\,dx\,dy \\ &= \int_0^1 \int_0^{2x^2} (x+y)\,dy\,dx \\ &= \int_0^1 \left(xy + \frac{y^2}{2} \Bigg|_{y=0}^{y=2x^2} \right) dx \\ &= \int_0^1 (2x^3 + 2x^4)\,dx \\ &= \frac{x^4}{2} + \frac{2x^5}{5} \Bigg|_0^1 = \frac{9}{10}. \end{aligned}$$

The moments are

$$\begin{aligned} M_x &= \iint\limits_R y\delta(x,y)\,dx\,dy \\ &= \int_0^1 \int_0^{2x^2} y(x+y)\,dy\,dx \\ &= \int_0^1 \left(\frac{xy^2}{2} + \frac{y^3}{3} \Bigg|_{y=0}^{y=2x^2} \right) dx \\ &= \int_0^1 (2x^5 + \frac{8x^6}{3})\,dx \\ &= \frac{x^6}{3} + \frac{8x^7}{21} \Bigg|_0^1 = \frac{5}{7} \end{aligned}$$

$$\begin{aligned} M_y &= \iint\limits_R x\delta(x,y)\,dx\,dy \\ &= \int_0^1 \int_0^{2x^2} x(x+y)\,dy\,dx \\ &= \int_0^1 \left(x^2y + \frac{xy^2}{2} \Bigg|_{y=0}^{y=2x^2} \right) dx \\ &= \int_0^1 (2x^4 + 2x^5)\,dx \\ &= \frac{2x^5}{5} + \frac{x^6}{3} \Bigg|_0^1 = \frac{11}{15}\,, \end{aligned}$$

so the center of mass $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{M_y}{M} = \frac{11/15}{9/10} = \frac{22}{27}\,, \quad \bar{y} = \frac{M_x}{M} = \frac{5/7}{9/10} = \frac{50}{63}\,.$$

Note how this center of mass is a little further towards the upper corner of the region R than when the density is uniform (it is easy see that for a uniform density we have $(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{3}{5}\right)$). This makes sense since the density function $\delta(x,y) = x + y$ increases as (x, y) approaches that upper corner.

6.12 Center of Mass in $\mathbb{R}^3$

The formulas for the center of mass of a region in $\mathbb{R}^2$ can be generalized to a solid S in $\mathbb{R}^3$. Let S be a solid with a continuous mass density function $\delta(x, y, z)$ at any point

(x, y, z) in S. Then the center of mass of S has coordinates $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} \;=\; \frac{M_{yz}}{M} \;, \quad \bar{y} \;=\; \frac{M_{xz}}{M} \;, \quad \bar{z} \;=\; \frac{M_{xy}}{M} \;, \tag{6.13}$$

where

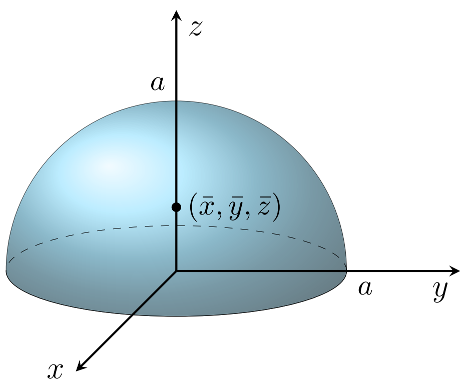
$$M \;=\; \iiint\limits_S \delta(x, y, z)\,dx\,dy\,dz \;,$$

is the *mass* of S, and

$$\begin{aligned} M_{yz} \;&=\; \iiint\limits_S x\delta(x, y, z)\,dx\,dy\,dz \;, \\ M_{xz} \;&=\; \iiint\limits_S y\delta(x, y, z)\,dx\,dy\,dz \;, \\ M_{xy} \;&=\; \iiint\limits_S z\delta(x, y, z)\,dx\,dy\,dz. \end{aligned}$$

In this case, M_{yz}, M_{xz} and M_{xy} are called the *moments* (or *first moments*) of S around the yz-plane, xz-plane and xy-plane, respectively.

Example 6.22. Find the center of mass of the solid $S = \{(x, y, z) : z \ge 0,\ x^2 + y^2 + z^2 \le a^2\}$, if the density function at (x, y, z) is $\delta(x, y, z) = 1$.



The solid S is just the upper hemisphere inside the sphere of radius a centered at the origin. Since the density function is a constant and S is symmetric about the z-axis, it is clear that $\bar{x} = 0$ and $\bar{y} = 0$, so we need only find $\bar{z}$. We have

$$M \;=\; \iiint\limits_S \delta(x, y, z)\,dx\,dy\,dz = \iiint\limits_S 1\,dx\,dy\,dz = \text{Volume}(S).$$

But since the volume of S is half the volume of the sphere of radius a, which we know is $\frac{4\pi a^3}{3}$, we conclude $M = \frac{2\pi a^3}{3}$. And

$$\begin{aligned}
M_{xy} &= \iiint\limits_S z\delta(x,y,z)\,dx\,dy\,dz \\
&= \iiint\limits_S z\,dx\,dy\,dz \quad \text{, which in spherical coordinates is} \\
&= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (r\,\cos\phi)\,r^2\,\sin\phi\,dr\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \sin\phi\,\cos\phi\left(\int_0^a r^3\,dr\right)\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \tfrac{a^4}{4}\,\sin\phi\,\cos\phi\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \tfrac{a^4}{8}\,\sin 2\phi\,d\phi\,d\theta \quad (\text{since } \sin 2\phi = 2\sin\phi\,\cos\phi) \\
&= \int_0^{2\pi}\left(-\tfrac{a^4}{16}\,\cos 2\phi\,\Big|_{\phi=0}^{\phi=\pi/2}\right)\,d\theta \\
&= \int_0^{2\pi}\tfrac{a^4}{8}\,d\theta \\
&= \frac{\pi a^4}{4}.
\end{aligned}$$

I conclusion, we have

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\frac{\pi a^4}{4}}{\frac{2\pi a^3}{3}} = \frac{3a}{8}\ .$$

Thus, the center of mass of S is $(\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{3a}{8}\right)$.

Chapter 7

Ordinary Differential Equations

7.1 Introduction to First Order Differential Equations

Definition 7.1 (First Order Differential Equation)**.** Let $I \subseteq \mathbb{R}$ be an open interval and $F\colon I \times \mathbb{R} \to \mathbb{R}$ a continuous function in two variables. An equation of the form

$$y'(x) = F(x, y(x)) \tag{7.1}$$

is called a *first order differential equation*.

If J is an open sub-interval of I, then a function $y\colon J \to \mathbb{R}$ of class $C^1(J)$ is called a *solution* to this differential equation if $y'(x) = F(x, y(x))$ holds for all $x \in J$. Sometimes we refer to y also as a *solution on* J to emphasize that the domain of y is different (i.e. smaller) than the domain of the differential equation.

In differential equations, it is common to write the function $y(x)$ without explicitly denoting its dependence on the variable x. This is known as the standard notation for differential equations. Using this notation, the first-order differential equation introduced in (7.1) becomes

$$y' = F(x, y), \tag{7.2}$$

where now it is implicitly understood that y is a function of x. In this context, we refer to y as the *dependent variable*, and to x as the *independent variable*. Sometimes, we will use t instead of x for the independent variable and u, v, or w instead of y for the dependent variable.

Example 7.1. Here are some examples of first order differential equations:

Equation	**Solution**
$y' = 2x$	$y = x^2 + c, \quad c \in \mathbb{R}$
$y' = -y$	$y = c\,e^{-x}, \quad c \in \mathbb{R}$
$tu'(t) + u(t) = 2t$	$u(t) = t + \frac{1}{t}$
$y' = \frac{1}{2}(y^2 - 1)$	$y = \frac{1+e^x}{1-e^x}$

Definition 7.2 (First Order Initial Value Problem)**.** Let $I \subseteq \mathbb{R}$ be an open interval, $F\colon I \times \mathbb{R} \to \mathbb{R}$ a continuous function, and $(x_0, y_0) \in I \times \mathbb{R}$. A *first order initial value problem* consists of two equations, a first order differential equation and an equation called the *initial condition*:

$$\underbrace{y' = F(x,y)}_{\text{differential equation}} \qquad \text{and} \qquad \underbrace{y(x_0) = y_0}_{\text{initial condition}} .$$

The point (x_0, y_0) is called the *initial value*, and the purpose of the initial condition is to specify the value of the solution to the differential equation at the initial value.

Given an open sub-interval J of I with $x_0 \in J$, a function $y\colon J \to \mathbb{R}$ is called a *solution* to this initial value problem if it is a solution to the differential equation $y' = F(x,y)$ and additionally the graph of y goes through the point (x_0, y_0). Occasionally, we will also refer to y as a *solution on* J to stress that the domain of y is equal to J and not I.

7.2 Antiderivatives as Differential Equations

The simplest type of differential equation are equations of the form

$$y' = f(x) \tag{7.3}$$

where $f\colon (a,b) \to \mathbb{R}$ is a continuous function. Recall the *Fundamental Theorem of Calculus*, which says that the function

$$F(x) = \int_a^x f(t)\,dt$$

is continuously differentiable on (a,b), it is an antiderivative of f, which means

$$F'(x) = f(x),$$

and all antiderivaties of f are of the form $F(x) + c$ for $c \in \mathbb{R}$. This allows us to completely describe all solutions to the differential equation (7.3).

Theorem 7.1. *Let $f\colon (a,b) \to \mathbb{R}$ be a continuous function. The set of solutions to the differential equation*

$$y' = f(x)$$

is precisely the set of all antiderivatives of f, i.e., all functions of the form $y(x) = \int_a^x f(t) + c$ for $c \in \mathbb{R}$.

Corollary 7.1. *Given $(x_0, y_0) \in (a, b) \times \mathbb{R}$, the initial value problem*

$$y' = f(x) \qquad \text{and} \qquad y(x_0) = y_0$$

has a unique solution given by $y(x) = \int_{x_0}^x f(t)\,dt + y_0$.

Example 7.2. A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of 10 m/s, and the only force acting on it is gravity. The ball has a mass of 0.15 kg at Earth's surface. Find the velocity $v(t)$ of the baseball at t seconds after the throw.

The differential equation that applies in this situation is

$$v'(t) = -g,$$

where $g = 9.8\,\mathrm{m/s}^2$. The initial condition is $v(0) = v_0$, where $v_0 = 10\,\mathrm{m/s}$. Therefore, the initial-value problem is

$$v'(t) = -9.8\,\mathrm{m/s}^2 \qquad \text{and} \qquad v(0) = 10\,\mathrm{m/s}.$$

In light of Corollary 7.1, there exists a unique solution to this intial value problem and it is given by

$$v(t) = \int_0^t v'(t)\,dt + v_0.$$

Since $v'(t) = -9.8$ and $v_0 = 10$, we get $v(t) = -9.8t + 10$.

7.3 A Toy Example

Consider an open interval $I \subseteq \mathbb{R}$ and a continuous function $p\colon I \to \mathbb{R}$. We aim to determine all solutions $u\colon I \to \mathbb{R}$ to the differential equation

$$u'(t) + p(t)u(t) = 0.$$

This equation is classified both as a *first order separable differential equation* and as a *first order linear differential equation*. We will discuss first order separable differential equations and first order linear differential equation in more detail in the upcoming sections.

Method 1: To solve the equation $u'(t) + p(t)u(t) = 0$, assume $u\colon I \to \mathbb{R}$ is a solution and choose $t_0 \in I$. Furthermore, assume that u does not vanish at any point in I. Then, for every $t \in I$,

$$\frac{u'(t)}{u(t)} = -p(t) \implies (\ln|u(t)|)' = -p(t) \implies \ln|u(t)| = -\int_{t_0}^t p(s)\,ds + C,$$

where $a \in \mathbb{R}$ is a constant. Thus, it is necessary that $|u(t)| = e^a e^{-\int_{t_0}^t p(s)\,ds}$ and finally that $u\colon I \to \mathbb{R}$ takes the form

$$u(t) = ce^{-\int_{t_0}^t p(s)\,ds}, \quad t \in J, \tag{7.4}$$

where $c \in \mathbb{R}\backslash\{0\}$ is a non-zero constant. Conversely, we can verify that for any constant $c \in \mathbb{R}\backslash\{0\}$ and any $t_0 \in I$, the function defined by (7.4) is a solution to the differential equation. Indeed,

$$u'(t) = ce^{-\int_{t_0}^t p(s)\,ds}\left(-\int_{t_0}^t p(s)\,ds\right)' = ce^{-\int_{t_0}^t p(s)\,ds}(-p(t)) = -p(t)u(t).$$

Moreover, when $c = 0$ the function u given by (7.4) vanishes throughout I and is also a solution.

Method 2: The first method does not exclude the possibility of a solution that vanishes at some point in I. To complete the discussion, let us assume again that $u\colon I \to \mathbb{R}$ is a solution and choose $t_0 \in J$, but this time we allow solutions u that may vanish at some points in I. Then, for every $t \in I$,

$$\begin{aligned} e^{\int_{t_0}^t p(s)\,ds}u'(t) + e^{\int_{t_0}^t p(s)\,ds}p(t)u(t) &= 0, \\ \implies \left(e^{\int_{t_0}^t p(s)\,ds}u(t)\right)' &= 0, \end{aligned} \tag{7.5}$$

and therefore, there exists a constant $c \in \mathbb{R}$ such that $e^{\int_{t_0}^t p(s)\,ds}u(t) = c$. We conclude that also in this case the solution $u(t)$ takes the form (7.4). This second method provides a complete resolution. Indeed, letting $q(t) = e^{\int_{t_0}^t p(s)\,ds}$, the second method exploids the fact that the two differential equations

$$\begin{gathered} u'(t) + p(t)u(t) = 0 \\ (u(t)q(t))' = 0 \end{gathered}$$

are equivalent, by which we mean that they have the same solutions. Since every solution to our differetnial equation is of the form

$$u(t) = ce^{-\int_{t_0}^t p(s)\,ds}, \quad t \in I, \quad c \text{ real constant}, \tag{7.6}$$

we say that (7.6) is the general solution. It follows that the initial value problem

$$u'(t) + p(t)u(t) = 0 \qquad \text{and} \qquad u(t_0) = u_0$$

has a unique solution given by

$$u(t) = u_0 e^{-\int_{t_0}^t p(s)\,ds}, \quad t \in I.$$

In Method 1, we solved the differential equation using an approach called “separation of variables”, which is the method used for solving separable differential equations. We will learn the general theory behind this method in the upcoming section. In Method 2, we multiplied the left side of the differential equation $u' + pu = 0$ by the function $e^{\int_{t_0}^{t} p(s)\,ds}$, which is called an *integrating factor*. This method is used for solving linear differential equations and we will explore this approach in more detail later too.

7.4 Maximal solutions to Differential Equations

Definition 7.3 (Maximal Solution to Differential Equation)**.** Let $I \subseteq \mathbb{R}$ be an open interval, $F\colon I \times \mathbb{R} \to \mathbb{R}$ a continuous function, and consider the first order differential equation

$$y'(x) = F(x, y(x)). \tag{7.7}$$

We say that a solution $y\colon J_{max} \to \mathbb{R}$ on an open interval $J_{max} \subseteq I$ is a *maximal solution* to the differential equation (7.7) if it is not a restriction of any other solution whose domain is a larger interval than J_{max}. In this context, we refer to the open interval J_{max} as the *maximal interval* corresponding to the maximal solution y.

Definition 7.4 (Maximal Solution to Initial Value Problem)**.** Let $I \subseteq \mathbb{R}$ be an open interval, $F\colon I \times \mathbb{R} \to \mathbb{R}$ a continuous function, and consider the first order initial value problem

$$y'(x) = F(x, y(x)) \qquad \text{and} \qquad y(x_0) = y_0 \tag{7.8}$$

for $(x_0, y_0) \in I \times \mathbb{R}$. We say that a solution $y\colon J_{max} \to \mathbb{R}$ on an open interval $J_{max} \subseteq I$ is a *maximal solution* to this initial value problem if it is a maximal solution to the differential equation $y'(x) = F(x, y(x))$ that also solves the initial value problem $y(x_0) = y_0$.

Example 7.3. Let $I \subseteq \mathbb{R}$ be an open interval and consider the differential equation

$$u'(t) + p(t)u(t) = 0,$$

where $p\colon I \to \mathbb{R}$ is a continuous function. As we have seen in Section 7.3, all solutions to this equation are of the form

$$u(t) = ce^{-\int_{t_0}^{t} p(s)\,ds},$$

where $c \in \mathbb{R}$, and the domain of these solutions is the same as the domain of $p(t)$. Hence all these solutions are maximal solutions and the maximal interval always equals I.

Clearly, it is not possible for the maximal interval J_{max} of a maximal solution to be bigger than the interval I, the domain of the differential equation. As we have seen in Example 7.3, it is possible that the maximal interval J_{max} coincides with the interval I. However, this is not always the case. For an example where the maximal intervals J_{max} of the maximal solutions are strictly smaller than the domain interval I, see Example 7.7 below.

7.5 First Order Separable Differential Equations

Definition 7.5. A differential equation in the form

$$y'h(y) = g(x) \tag{7.9}$$

where $g\colon I_1 \to \mathbb{R}$ is a continuous function over the open interval I_1, and $h\colon I_2 \to \mathbb{R}$ is a continuous function over the open interval I_2, is called a *(first order) separable differential equation.*

The name "separable" comes from the fact that the expression on the right side only involves the dependent variable y whereas the expression on the left side only involves the independent variable x, so the variables are separated from one another on different sides of the equation. Any first order differential equation that can we rearranged such that the variables are separated is a first order separable differential equation, see items 2 and 3 in the following example.

Example 7.4 (Examples of separable differential equations)**.**

1. In Section 7.2 we discussed differential equations of the form $y' = f(x)$. Any such differential equation is clearly separable because it has the form $y'h(y) = g(x)$. Indeed,
$$y' = f(x) \quad \text{with} \quad \begin{cases} g(x) = f(x), \\ h(y) = 1. \end{cases}$$
2. The differential equation $y' + y^2\cos(2t) = 0$ is separable, since it can be written as
$$\frac{y'}{y^2} = -\cos(2t) \quad \text{with} \quad \begin{cases} g(t) = -\cos(2t), \\ h(y) = \frac{1}{y^2}. \end{cases}$$
3. In Section 7.3 we discussed differential equations of the form $u'(t)+p(t)u(t) = 0$. Any differential equation of this form is separable, because it can we written as
$$\frac{u'(t)}{u(t)} = -p(t) \quad \text{with} \quad \begin{cases} g(t) = -p(t), \\ h(y) = \frac{1}{u(t)}. \end{cases}$$
4. The equation $y' = e^y + \cos(t)$ is not separable.
5. Consider the differential equation $y' = ay + b(t)$, where $a \in \mathbb{R}\backslash\{0\}$ and $b(t)$ is a continuous function. If $b(t)$ is non-constant then this differential equation is not separable. On the other hand, if $b(t) = b$ is constant then the differential equation is separable, because we can write it as
$$\frac{y'}{ay+b} = 1 \quad \text{with} \quad \begin{cases} g(t) = 1, \\ h(y) = \frac{1}{ay+b}. \end{cases}$$

Method of Separation of Variables: To solve a separable differential equation, we us the method of separation of variables. We start by expressing $y' = \frac{dy}{dx}$ in Leibniz notation and rewrite the equation in differential form as

$$h(y)\,dy = g(x)\,dx.$$

Then we integrate both sides of the equation and obtain an integral equation of the form

$$\int h(y)\,dy = \int g(x)\,dx.$$

If $H(y)$ is an antiderivative of $h(y)$ and $G(x)$ is an antiderivative of $g(x)$, then we find that

$$H(y) = G(x) + c,$$

where c is a constant. This last equation defines y implicitly as a function of x. In some cases, we may be able to solve for y in terms of x and thus find the solution to the differential equation, but this is not always possible.

Definition 7.6. Let $H\colon I_2 \to \mathbb{R}$ and $G\colon I_1 \to \mathbb{R}$ be continuous functions and $c \in \mathbb{R}$. If J is a open sub-interval of I_1, then we say that a function $y\colon J \to I_2$ is a *solution* to the implicit (functional) equation

$$H(y) = G(x) + c$$

if $H(y(x)) = G(x)$ holds for all $x \in J$.

Theorem 7.2. *Suppose $g\colon I_1 \to \mathbb{R}$ and $h\colon I_2 \to \mathbb{R}$ are continuous functions, and J is an open sub-interval of I_1. A function $y\colon J \to I_2$ of class $C^1(J)$ is a solution on J to the differential equation*

$$y'h(y) = g(x)$$

if and only if it is a solution on J to the implicit equation

$$H(y) = G(x) + c$$

for some $c \in \mathbb{R}$, where $G\colon I_1 \to \mathbb{R}$ and $H\colon I_2 \to \mathbb{R}$ are antiderivatives of g and h, respectively. In the case that the function H is invertible on the set $\{G(x)+c : x \in J\}$, the solution to the differential equation can be given in explicit form as

$$y(x) = H^{-1}(G(x) + c).$$

Finding Explicit Solutions to Seperable Equations. Needless to say, we prefer explicit solutions over implicit solutions. Unfortunately, Theorem 7.2 tells us that our capability to find explicit solutions depends on whether $H(y)$, the antiderivative of $h(y)$, is invertible. In practice, there are three possible scenarios that can occur:

- The inverse function H^{-1} can be found algebraically by solving for the input variable in terms of the output variable. See Example 7.5 for an illustration of this case.

- The inverse function H^{-1} cannot be expressed in a closed algebraic form. See Example 7.6 for a separable differential equation of this nature.
- The function H has multiple branches, leading to multiple inverse functions. In this case each inverse function leads to an explicit solution for the differential equation. See Example 7.7 for an instance where this occurs.

Example 7.5. Let us solve the differential equation

$$y' = \frac{x^2}{y^2}.$$

Using the method of separation of variables, we separate the variables, write the equation in terms of differentials, and integrate both sides:

$$\begin{aligned} y^2\, dy &= x^2\, dx \\ \int y^2\, dy &= \int x^2\, dx \\ \tfrac{1}{3}y^3 &= \tfrac{1}{3}x^3 + C, \end{aligned}$$

where C is an arbitrary constant. Note that the function $H(y) = \frac{1}{3}y^3$ is invertible on all of $\mathbb{R}$, and tts inverse function can be found through elementary algebraic manipulations by expressing the input variable in terms of the output variable. After completing this process, we get that the inverse function of $H(y) = \frac{1}{3}y^3$ is $H^{-1}(y) = \sqrt[3]{3y}$. This means we get the solution to the differential equation in explicit form as

$$y(x) = \sqrt[3]{x^3 + 3C}.$$

We could leave the solution like this or we could write it in the form

$$y = \sqrt[3]{x^3 + K},$$

where $K = 3C$ (since C is an arbitrary constant).

Example 7.6. Let us solve the differential equation

$$y' = \frac{6x^2}{2y + \cos y}.$$

This is a separable differential equation. Thus, using Theorem 7.2, we see that all the solutions to this differential equation are given by solutions to the implicit equation

$$y^2 + \sin y = 2x^3 + c,$$

where $c \in \mathbb{R}$ is a constant. This gives the general solution implicitly. In this case it's impossible express y explicitly as a function of x in a closed algebraic from, so we keep the implicit solution.

Example 7.7. Consider the differential equation

$$y' = -\frac{xy}{\ln y}.$$

This is a separable differential equation, so we can separate the variables and find an implicit solution. Indeed, we have

$$\begin{aligned}
\frac{\ln y}{y}\,dy &= -x\,dx \\
2\frac{\ln y}{y}\,dy &= -2x\,dx \\
\int 2\frac{\ln y}{y}\,dy &= -\int 2x\,dx \\
\ln(y)^2 &= -x^2 + c, \qquad c \in \mathbb{R}.
\end{aligned}$$

Note that the function $H\colon (0,\infty) \to (0,\infty)$, $H(y) = \ln(y)^2$, is not invertible on its entire domain, but it can be split into two branches such that it becomes invertible on each branch. The two branches are

$$\begin{aligned}
H\colon (0,1) \to (0,\infty), &\quad H(y) = \ln(y)^2, \\
H\colon (1,\infty) \to (0,\infty), &\quad H(y) = \ln(y)^2.
\end{aligned}$$

This leads to two inverse functions, one for each branch, given by

$$\begin{aligned}
H^{-1}\colon (0,\infty) \to (0,1), &\quad H^{-1}(y) = e^{-\sqrt{y}}, \\
H^{-1}\colon (0,\infty) \to (1,\infty), &\quad H^{-1}(y) = e^{\sqrt{y}}.
\end{aligned}$$

The reader can check that both of these functions are indeed inverse functions of H, simply by verifying that $H(H^{-1}(y)) = y$ is true for both of them. Thus for every $c > 0$ we get two explicit solutions to the differential equation, which are

$$\begin{aligned}
y\colon (-\sqrt{c}, \sqrt{c}) \to \mathbb{R}, &\quad y(x) = e^{\sqrt{c-x^2}}, \\
y\colon (-\sqrt{c}, \sqrt{c}) \to \mathbb{R}, &\quad y(x) = e^{-\sqrt{c-x^2}}.
\end{aligned}$$

In fact, these are maximal solutions (cf. Definition 7.3) and the corresponding maximal intervals are $(-\sqrt{c}, \sqrt{c})$.

Corollary 7.2. *With the same notation as in Theorem 7.2. Let $(x_0, y_0) \in I_1 \times I_2$. Then $y\colon J \to I_2$ is a solution to the initial value problem*

$$y'h(y) = g(x) \qquad \text{and} \qquad y(x_0) = y_0$$

if and only if it is a solution to the implicit equation

$$H(y) = G(x) - G(x_0) + H(y_0).$$

In the case that the function H is invertible on the set $\{G(x) - G(x_0) + H(y_0) : x \in J\}$, *the solution to the initial value problem can be given in explicit form as*

$$y(x) = H^{-1}(G(x) - G(x_0) + H(y_0)).$$

Example 7.8. Consider the initial value problem:

$$\frac{u'(t)}{u^2(t)} = 1 \qquad \text{and} \qquad u(0) = u_0 \neq 0.$$

This differential equation is separable, because one may choose $I_1 = \mathbb{R}$ and $I_2 = (0, +\infty)$ or $I_2 = (-\infty, 0)$, depending on whether $u_0 > 0$ or $u_0 < 0$, and define $g\colon \mathbb{R} \to \mathbb{R}$ and $h\colon I_1 \to \mathbb{R}$ by

$$h(u) = \frac{1}{u^2} \qquad \text{and} \qquad g(t) = 1.$$

Since this is a separable differential equation, the solutions are given by the implicit equation

$$-\frac{1}{u(t)} = t + c, \qquad c \in \mathbb{R}.$$

Under the assumption $t + c \neq 0$, this can be turned into an explicit equation and we get

$$u(t) = -\frac{1}{c+t}, \qquad c \in \mathbb{R}.$$

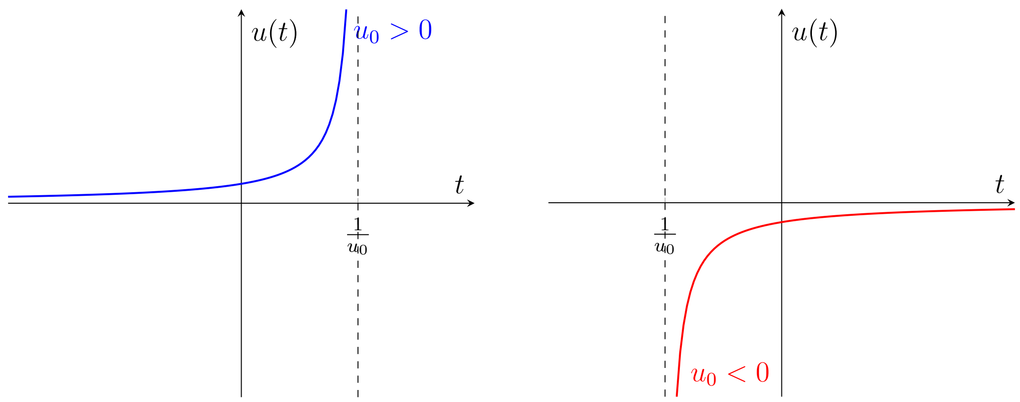
The initial condition $u(0) = u_0$ is satisfied when $c = -\frac{1}{u_0}$ and hence

$$u(t) = \frac{u_0}{1 - u_0 t}.$$

The maximal solutions to the initial value problem can then be described as

$$\begin{cases} u(t) = \frac{u_0}{1-u_0 t} \text{ with maximal interval } t \in (-\infty, \frac{1}{u_0}), & \text{if } u_0 > 0, \\ u(t) = \frac{u_0}{1-u_0 t} \text{ with maximal interval } t \in (\frac{1}{u_0}, \infty), & \text{if } u_0 < 0. \end{cases}$$

This example shows that, in general, the maximal interval can also depend on the domain of h.



Theorem 7.3 (Existence and uniqueness theorem). *With the same notation as in Theorem 7.2 and Corollary 7.2, consider the initial value problem*

$$y'h(y) = g(x) \qquad \text{and} \qquad y(x_0) = y_0.$$

If $h(y_0) \neq 0$ then there exists an open interval J_{max} containing x_0 and a function $y\colon J_{max} \to I_2$ such that y is a maximal solution to the above initial value problem. Moreover, this solution is unique in the sense that if $\tilde{J}$ is another open interval containing x_0 and $\tilde{y}\colon \tilde{J} \to I_2$ is another solution to the above initial value problem then $\tilde{J} \subseteq J_{max}$ and $y(x) = \tilde{y}(x)$ holds for all $x \in \tilde{J}$.

The next example shows that if the condition $h(y_0) \neq 0$ in Theorem 7.3 is not satisfied then the initial value problem might not have any solutions at all.

Example 7.9 (Non-existence of solution). The initial value problem

$$u(t)u'(t) = 1 \qquad \text{and} \qquad u(0) = 0$$

admits no solution. Indeed, if a solution $u : J \to \mathbb{R}$ would exist over a certain open interval J containing $t = 0$, it would lead to the contradiction $0 = 0 \cdot u'(0) = u(0)u'(0) = 1$. Although $u(t) = \sqrt{2t}$ is a solution to the differential equation on $(0, +\infty)$, $t = 0$ is not in the domain of definition of this solution.

In contrast to Example 7.9, the next example shows that if the condition $h(y_0) \neq 0$ in Theorem 7.3 is not satisfied then there might be solutions, but they might not be unique.

Example 7.10 (Non-uniqueness of solution). Consider the initial value problem

$$u(t)u'(t) = t \qquad \text{and} \qquad u(0) = 0.$$

This problem has two solutions for the same initial condition, namely $u(t) = t$ and $u(t) = -t$. We conclude that even though solutions exist, they are not unique.

Example 7.11. Let $n \geqslant 2$ be an integer, $I_1 \subseteq \mathbb{R}$ an open interval, $I_2 = (0, \infty)$, $(t_0, u_0) \in I_1 \times I_2$, and $g\colon I_1 \to \mathbb{R}$ a continuous function. Consider the separable initial value problem

$$\frac{u'(t)}{u^n(t)} = g(t) \qquad \text{and} \qquad u(t_0) = u_0.$$

In view of Theorem 7.2, the solution to the above differential equation is given by the solution to the implicit equation

$$\frac{u^{-n+1}(t)}{-n+1} = \int_{t_0}^{t} g(s)\,ds + c.$$

Since $u(t) \in I_2 = (0, \infty)$ and $-n + 1 < 0$, we have $\int_{t_0}^{t} g(s)\,ds + c < 0$. Hence,

$$u(t) = \left((1-n)\left(c + \int_{t_0}^{t} g(s)\,ds\right)\right)^{\frac{1}{1-n}}.$$

For the initial condition $u(t_0) = u_0 > 0$, we thus necessarily obtain

$$u(t) = \left(u_0^{1-n} + (1-n)\int_{t_0}^{t} g(s)\,ds\right)^{\frac{1}{1-n}} = u_0\left(1 + (1-n)u_0^{n-1}\int_{t_0}^{t} g(s)\,ds\right)^{\frac{1}{1-n}}, \quad (7.10)$$

with $u_0^{1-n} + (1-n)\int_{t_0}^{t} g(s)\,ds > 0$. Conversely, under the encountered restrictions, the unique maximal solution $u\colon J_{max} \to I_2$ is thus

$$u(t) = u_0 \cdot \left(1 + (1-n)u_0^{n-1}\int_{t_0}^{t} g(s)\,ds\right)^{\frac{1}{1-n}},$$

where J_{max} is the largest sub-interval of $\{t \in I_1 : 1 + (1-n)u_0^{n-1}\int_{t_0}^{t} g(s)\,ds > 0\}$ containing t_0. Consequently, one can verify that if $J_{max} = (a,b)$ for $a, b \in \mathbb{R} \cup \{\pm\infty\}$ and $a \neq -\infty$ (respectively $b \neq +\infty$) then

$$\lim_{t\to a^+} u(t) = +\infty \quad \text{(respectively } \lim_{t\to b^-} u(t) = +\infty \text{).}$$

Remark 7.1. The case $I_2 = (-\infty, 0)$ and $u_0 < 0$ is handled analogously.

7.6 First Order Homogeneous Differential Equations

Occasionally, a differential equation may not initially be separable, but it can be converted into a separable equation by altering the unknown function. This situation arises with a class of differential equations termed Euler homogeneous equations.

Definition 7.7. An *Euler homogeneous equation* is a first order differential equation of the form

$$u'(t) = F\left(\frac{u(t)}{t}\right) \quad (7.11)$$

where $F\colon I \to \mathbb{R}$ is a continuous function on an open interval I in $\mathbb{R}$.

Any function of the from $H(t,u) = F(\frac{u}{t})$ that depends only on the quotient $\frac{y}{t}$ is scale invariant. This means that it does not change under the transformation $u \to cu$ and $t \to ct$, i.e.,

$$H(cu, ct) = H(u, t).$$

For this reason, the differential equations above are also called scale invariant equations.

Scale invariant functions are a particular case of homogeneous functions, which we define now.

Definition 7.8. A real-valued function $N\colon \mathbb{R}^2 \to \mathbb{R}$ is called *homogeneous of degree n* if it satisfies $N(cx, cy) = c^n N(x,y)$ for all $c \in \mathbb{R}$.

Example 7.12 (Examples of homogeneous functions of various degrees)**.**

1. Any function of the form $N(x, y) = F(\frac{y}{t})$ is homogeneous of degree 0. This fact underlines the connection between homogeneous functions and Euler homogeneous differential equations.
2. The function $N(x, y) = x + 4y$ is homogeneous of degree 1.
3. The function $N(x, y) = xy$ is homogeneous of degree 2.
4. A "real-life" example of a homogeneous function is the energy of a thermodynamic system, such as a gas in a bottle. The energy E of a fixed amount of gas is a function of the gas entropy S and the gas volume V. Such energy is a homogeneous function of degree one, i.e., $E(cS, cV) = cE(S, V)$ for all $c \in \mathbb{R}$.

Euler homogeneous differential equations often arise from differential equations of the from

$$N(t, y(t))y'(t) + M(t, y(t)) = 0,$$

where both N and M are homogeneous functions of the same degree. Indeed, we can rewrite this differential equations as

$$y'(t) = -\frac{M(t, y)}{N(t, y)}.$$

If we now define

$$F(x) = -\frac{M(1, x)}{N(1, x)}$$

then we see that $-\frac{M(t,y)}{N(t,y)} = F(\frac{y}{t})$ and hence the above differential equation is equivalent to

$$y'(t) = F\left(\frac{y(t)}{t}\right),$$

which is Euler homogeneous.

Theorem 7.4. *If the functions $N\colon \mathbb{R}^2 \to \mathbb{R}$ and $M\colon \mathbb{R}^2 \to \mathbb{R}$ are homogeneous of the same degree, then the differential equation*

$$N(t, y)y' + M(t, y) = 0$$

is Euler homogeneous as it is equivalent to the differential equation

$$y' = F\left(\frac{y}{t}\right)$$

where $F(x) = -\frac{M(1,x)}{N(1,x)}$.

Our next aim is to discuss how to solve Euler homogeneous differential equations.

The Substitution Method: To solve homogeneous differential equations, we will use the substitution method to transform them into separable differential equations.

The substitution that we will utilize is given by the equation

$$u(t) = tv(t). \tag{7.12}$$

Differentiation this equation gives the substitution for $u'(t)$:

$$u'(t) = v(t) + tv'(t). \tag{7.13}$$

Taking (7.12) and (7.13) and substituting them into (7.11), we obtain a new equivalent differential equation,

$$v(t) + tv'(t) = F(v(t)).$$

Provided that, for every $x \in I$ we have $F(x) \neq x$, we can rearrange this new differential equation and get

$$\frac{1}{F(v(t)) - v(t)} v'(t) = \frac{1}{t}.$$

We have now arrived at a separable differential equation, which we already know how to solve it.

Theorem 7.5. *Let I be an open interval in $\mathbb{R}$ not containing 0 and suppose $F\colon I \to \mathbb{R}$ is a continuous function satisfying $F(x) \neq x$ for all $x \in I$. A function $v\colon J \to \mathbb{R}$ is a solution to the separable differential equation*

$$\frac{1}{F(v(t)) - v(t)} v'(t) = \frac{1}{t}$$

if and only if $u\colon J \to \mathbb{R}$ defined by $u(t) = tv(t)$ is a solution to the Euler homogeneous equation

$$u'(t) = F\left(\frac{u(t)}{t}\right).$$

Remark 7.2. The original homogeneous equation for the function u is transformed into a separable equation for the unknown function $v = \frac{u}{t}$. One solves for v, in implicit or explicit form, and then transforms back to $u = tv$.

Corollary 7.3. *Let I be an open interval in $\mathbb{R}$ not containing 0 and suppose $F\colon I \to \mathbb{R}$ is a continuous function satisfying $F(x) \neq x$ for all $x \in I$. Let $(t_0, u_0) \in I \times \mathbb{R}$. A function $v\colon J \to \mathbb{R}$ is a solution to the separable initial value problem*

$$\frac{1}{F(v(t)) - v(t)} v'(t) = \frac{1}{t} \qquad \text{and} \qquad v(t_0) = \frac{u_0}{t_0}$$

if and only if $u\colon J \to \mathbb{R}$ defined by $u(t) = tv(t)$ is a solution to the Euler homogeneous initial value problem

$$u'(t) = F\left(\frac{u(t)}{t}\right) \qquad \text{and} \qquad u(t_0) = u_0.$$

Example 7.13. Let us find all solutions y of the differential equation $y' = \frac{t^2+3y^2}{2ty}$. According to Theorem 7.4, the equation is Euler homogeneous, since

$$M(ct, cy) = \frac{c^2t^2 + 3c^2y^2}{2(ct)(cy)} = \frac{c^2(t^2 + 3y^2)}{c^2(2ty)} = \frac{t^2 + 3y^2}{2ty} = M(t, y).$$

Note that if $F(x) = \frac{1+3x^2}{2x}$ then

$$F\Big(\frac{y}{t}\Big) = \frac{1 + 3\left(\frac{y}{t}\right)^2}{2\left(\frac{y}{t}\right)} = \frac{t^2 + 3y^2}{2ty},$$

so we can rewrite the equation as

$$y' = F\Big(\frac{y}{t}\Big).$$

Now we introduce the change of functions $v = \frac{y}{t}$, and hence

$$y' = \frac{1 + 3v^2}{2v}.$$

Since $y = tv$, then $y' = v + tv'$, which implies

$$v + tv' = \frac{1 + 3v^2}{2v} \implies tv' = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v} = \frac{1 + v^2}{2v}.$$

We obtained the separable equation

$$v' = \frac{1}{t}\left(1 + \frac{v^2}{2}\right).$$

We rewrite and integrate it,

$$\frac{2v}{1 + v^2}v' = \frac{1}{t} \implies \int \frac{2v}{1 + v^2}dv = \int \frac{1}{t}dt + c_0.$$

The substitution $u = 1 + v^2$ implies $du = 2vv'dv$, so

$$\int \frac{du}{u} = \int \frac{dt}{t} + c_0 \implies \ln(u) = \ln(t) + c_0 \implies u = e^{\ln(t)+c_0}.$$

But $u = e^{\ln(t)}e^{c_0}$, so denoting $c_1 = e^{c_0}$, then $u = c_1t$. So, we get

$$1 + v^2 = c_1t \implies 1 + \Big(\frac{y}{t}\Big)^2 = c_1t \implies \Big(\frac{y}{t}\Big)^2 = c_1t - 1 \implies y(t) = \pm t\sqrt{c_1t - 1}.$$

Example 7.14. Consider the initial value problem

$$y' = \frac{x^3}{y^3} + \frac{y}{x}, \quad x > 0, \qquad \text{and} \qquad y(1) = -2.$$

The function $M(x, y) = \frac{x^3}{y^3} + \frac{y}{x}$ is homogeneous of order 0, which implies that we are dealing with an Euler homogeneous differential equation. Thus, we use the substitution

$y(x) = xv(x)$ to turn the given Euler homogeneous equation into a separable equation. Using

$$y(x) = xv(x) \qquad \text{and} \qquad y'(x) = v(x) + xv'(x),$$

we get

$$v + xv' = \frac{1}{v^3} + v \qquad \Longleftrightarrow \qquad v'v^3 = \frac{1}{x}.$$

Integrating both sides yields the implicit equation

$$\frac{v^4}{4} = \ln(x) + c, \qquad c \in \mathbb{R}.$$

The function $v \mapsto \frac{v^4}{4}$ has two branches on which it is invertible, the left branch $(-\infty, 0)$ and the right branch $(0, \infty)$. Each branch leads to a solution,

$$\begin{aligned} v\colon (e^{-c}, \infty) &\to (-\infty, 0), \quad v(x) = -\sqrt[4]{4\ln(x) + 4c}; \\ v\colon (e^{-c}, \infty) &\to (0, +\infty), \quad v(x) = \sqrt[4]{4\ln(x) + 4c}. \end{aligned}$$

Using $y(x) = xv(x)$, we obtain the general solutions to the initial differential equation,

$$\begin{aligned} y\colon (e^{-c}, \infty) &\to (-\infty, 0), \quad y(x) = -x\sqrt[4]{4\ln(x) + 4c}; \\ y\colon (e^{-c}, \infty) &\to (0, +\infty), \quad y(x) = x\sqrt[4]{4\ln(x) + 4c}. \end{aligned}$$

It remains to take into consideration the initial condition

$$y(1) = -2.$$

Since the y-value at the initial condition is negative, we must be in the left branch corresponding to the co-domain $(-\infty, 0)$. This means we use $y(x) = -x\sqrt[4]{4\ln(x) + 4c}$ and not $y(x) = x\sqrt[4]{4\ln(x) + 4c}$ to solve this inital value problem. Using the initial condition $y(1) = -2$, we determine that $c = 4$. Thus, the unique (and maximal) solution to the initial value problem is

$$y\colon (e^{-4}, \infty) \to (-\infty, 0), \quad y(x) = -x\sqrt[4]{4\ln(x) + 16}.$$

7.7 First Order Linear Differential Equations

Definition 7.9. Let $I \subseteq \mathbb{R}$ be an open interval and let $a\colon I \to \mathbb{R}$ and $b\colon I \to \mathbb{R}$ be continuous functions. A differential equation of the form

$$y'(t) = a(t)y(t) + b(t) \tag{7.14}$$

is called a *(first order) linear differential equation.* The equation (7.14) above:

(i) is *homogeneous* if the source function $b(t) = 0$ for all $t \in I$;
(ii) is *inhomogeneous* if the source function $b(t) \neq 0$ for some $t \in I$;

(iii) has *constant coefficient* if $a(t) = a$ for some constant $a \in \mathbb{R}$ and for all $t \in I$;
(iv) has *variable coefficient* if $a(t)$ is a non-constant function on I;

The name "linear" derives from the fact that the equation is a linear polynomial in the dependent variable y (i.e., it resembles the equation of a line $y \mapsto ay + b$). In accordance with this analogy, linear differential equations exhibit behavior reminiscent of the behavior of linear functions in algebra.

Case in point is the *principle of superposition*, which asserts that linear combinations[1] of solutions to linear differential equations are again solutions to linear differential equations. This implies that solutions can be combined and scaled to produce new solutions, simplifying the analysis of such differential equations and allowing for systematic methods of producing solutions.

Theorem 7.6 (Principal of Superposition)**.** *Let I be an open interval in $\mathbb{R}$, let $a, b_1, b_2 \colon I \to \mathbb{R}$ be three continuous functions and suppose the two functions $y_1, y_2 \colon I \to \mathbb{R}$ are respective solutions to the two linear differential equations*

$$y'(t) = a(t)y(t) + b_1(t) \qquad \text{and} \qquad y'(t) = a(t)y(t) + b_2(t).$$

Then for any $c_1, c_2 \in \mathbb{R}$ the function $y \colon I \to \mathbb{R}$ defined by

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

is a solution to the linear differential equation

$$y'(t) = a(t)y(t) + c_1b_1(t) + c_2b_2(t).$$

The Integrating Factor Method: To solve linear differential equations, we use what is called the integrating factor method. Let $A \colon I \to \mathbb{R}$ be an anitderivative of $a(t)$. The *integrating factor* associated to the linear differential equation

$$y'(t) = a(t)y(t) + b(t) \tag{7.15}$$

is the function $e^{-A(t)}$. Multiplying both sides of (7.15) by the integrating factor and rearranging, we obtain the equivalent differential equation

$$\underbrace{e^{-A(t)}y'(t) - e^{-A(t)}a(t)y(t)}_{\text{product rule}} = e^{-A(t)}b(t). \tag{7.16}$$

We notice that the left hand side equals the derivative of $e^{-A(t)}y(t)$ due to the product rule for differentiation. This means (7.16) can be simplified as

$$\left(e^{-A(t)}y(t)\right)' = e^{-A(t)}b(t). \tag{7.17}$$

[1]In physics and other sciences, the expression 'superposition' is often used as a synonym to 'linear combination'.

Letting t_0 be an arbitrary point in the interval I and integrating both sides of (7.17), we obtain

$$e^{-A(t)}y(t) = \int_{t_0}^{t} e^{-A(s)}b(s)\,ds + c, \qquad c \in \mathbb{R}.$$

Finally, isolating the term $y(t)$ we obtain the general solution to (7.15) as

$$y(t) \;=\; ce^{A(t)} + e^{A(t)}\int_{t_0}^{t} e^{-A(s)}b(s)\,ds, \qquad c \in \mathbb{R}.$$

Theorem 7.7. *Let $I \subseteq \mathbb{R}$ be an open interval, t_0 any element of I, and $a\colon I \to \mathbb{R}$ and $b\colon I \to \mathbb{R}$ two continuous functions. The solutions $y\colon I \to \mathbb{R}$ to the linear differential equation*

$$y'(t) = a(t)y(t) + b(t)$$

are given by

$$y(t) = \underbrace{ce^{A(t)}}_{\text{homogeneous part}} + \underbrace{e^{A(t)}\int_{t_0}^{t} e^{-A(s)}b(s)\,ds}_{\text{inhomogeneous part}}, \tag{7.18}$$

where $A\colon I \to \mathbb{R}$ is an antiderivative of $a(t)$ and c is any real number.

Remark 7.3. Here are some important observations regarding Theorem 7.7.

(a) The solutions provided in (7.18) are all maximal solutions with maximal interval I.

(b) Note that the expression $\int_{t_0}^{t} e^{-A(s)}b(s)$ in (7.18) describes an antiderivative of the function $e^{-A(t)}b(t)$. We can replace it by any other antiderivative of $e^{-A(t)}b(t)$. Thus, instead of (7.18), we could have also written

$$y(t) = ce^{A(t)} + e^{A(t)}D(t),$$

where $D\colon I \to \mathbb{R}$ is an antiderivative of $e^{-A(t)}b(t)$.

(c) The solution in (7.18) is comprised of two components. The first part $ce^{A(t)}$ is called the *homogeneous part* of the solution, and it is actually the general solution to the homogeneous linear differential equation

$$y'(t) = a(t)y(t). \tag{7.19}$$

The second part $e^{A(t)}\int_{t_0}^{t} e^{-A(s)}b(s)\,ds$ is called the *inhomogenous part*. It corresponds to the solution in (7.18) where $c = 0$, so it is one particular solution to the inhomogeneous linear differential equation

$$y'(t) = a(t)y(t) + b(t). \tag{7.20}$$

This means that if we have one particular solution to the inhomogeneous linear differential equation (7.20) then we can obtain all solutions by adding all the solutions to the corresponding homogeneous linear differential equation (7.19). This is in line with the Principal of Superposition (see Theorem 7.6), which tells us that the general solution to the inhomogeneous linear differential equation is obtained by taking the general solution of the homogeneous linear differential equation and adding to it a particular solution to the inhomogeneous linear differential equation.

Example 7.15. Let us find all solutions y to the differential equation

$$y'(t) = \frac{3}{t}y + t^5, \quad t > 0.$$

This is a linear differential equation with variable coefficient $a(t) = \frac{3}{t}$ and source function $b(t) = t^5$. To determine all solutions, we start by finding the antiderivative of $a(t)$:

$$A(t) = \int \tfrac{3}{s}\, ds = 3\ln(t).$$

Thus, the integrating factor for the linear differential equation at hand equals

$$e^{-A(t)} = e^{-3\ln(t)} = \frac{1}{t^3}.$$

Next, we need to find an antiderivative for $e^{-A(t)}b(t)$, i.e.,

$$D(t) = \int e^{-A(s)}b(s)\, ds = \int \frac{s^5}{s^3}\, ds = \int s^2\, ds = \frac{t^3}{3}.$$

It now follows from Theorem 7.7 (and part (b) of Remark 7.3) that the general solution to the given differential equation is

$$y(t) = ce^{A(t)} + e^{A(t)}D(t) = ct^3 + \frac{t^6}{3}.$$

where c is an arbitrary constant.

Corollary 7.4. *Let $I \subseteq \mathbb{R}$ be an open interval, and $a\colon I \to \mathbb{R}$ and $b\colon I \to \mathbb{R}$ two continuous functions. For any $(t_0, y_0) \in I \times \mathbb{R}$, the initial value problem*

$$y'(t) = a(t)y(t) + b(t) \qquad \text{and} \qquad y(t_0) = y_0$$

has a unique solution given by

$$y(t) = y_0 e^{A(t)-A(t_0)} + e^{A(t)} \int_{t_0}^{t} e^{-A(s)}b(s)\, ds,$$

where $A\colon I \to \mathbb{R}$ is an antiderivative of $a(t)$.

Example 7.16. A 50 liter tank of pure water has a brine mixture with a concentration of 2 grams per liter entering at the rate of 5 liters per minute (see Fig. 7.1.) At the same time, the well-mixed contents drain out at the rate of 5 liters per minute. Find the amount of salt in the tank at time t.

In all such problems, one assumes for simplicity that the solution is well mixed at each instant of time. Let $x(t)$ be the amount of salt at time t. Then the rate at which the salt in the tank increases is due to the amount of salt entering the tank less than

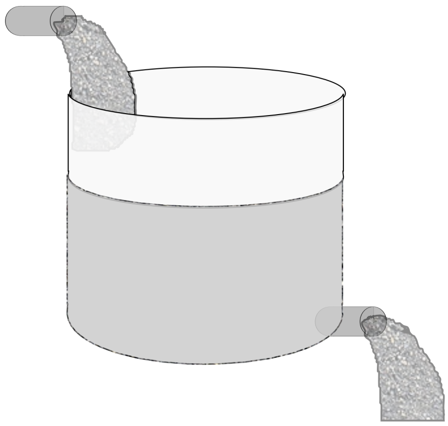


Figure 7.1: Illustration of 50 liter tank with brine mixture.

leaving the tank. To figure out these rates, one notes that $\frac{dx}{dt}$ has units of grams per minute. The amount of salt entering per minute is given by the product of the entering concentration times the rate at which the brine enters. This gives the correct units:

$$(2 \text{ grams/liter}) \times (5 \text{ liters/minute}) = 10 \text{ grams/minute}.$$

Similarly, one can determine the rate out as

$$\left(\frac{x}{50}\text{grams/liter}\right) \times (5 \text{ liters/minute}) = \frac{x}{10} \text{ grams/minute}.$$

Thus, we have the initial value problem

$$\frac{dx}{dt} = 10 - \frac{x}{10} \qquad \text{and} \qquad x(0) = 0,$$

which involves a first order linear differential equation. The integrating factor is $e^{x/10}$, leading to the general solution

$$x(t) = 100 + Ae^{-t/10}.$$

Using the initial condition, one finds the particular solution

$$x(t) = 100\left(1 - e^{-t/10}\right).$$

Often one is interested in the long-time behavior of a system. In this case, we have that $\lim_{t\to\infty} x(t) = 100$ grams. This makes sense because 2 grams per liter enter during this time to eventually leave the entire 50 liters with this concentration. Thus,

$$\frac{50 \text{ liters} \times 2 \text{ grams/liter}}{50 \text{ liters}} = 100 \text{ grams}.$$

Example 7.17. Find the solution to the initial value problem

$$ty' = -2y + 4t^2, \quad t > 0, \qquad \text{and} \qquad y(1) = -2.$$

First, we rewrite the equation as

$$y' = -\tfrac{2}{t}y + 4t,$$

and we notice that the differential equation is linear with variable coefficient $a(t) = -\frac{2}{t}$ and source function $b(t) = 4t$. Thus the integrating factor is

$$e^{-A(t)} = e^{2\ln(t)} = \left(e^{\ln(t)}\right)^2 = t^2,$$

where $A(t) = -2\ln(t)$ is an antiderivative of $a(t)$. Multiplying both sides of the differential equation with the integrating factor and rearranging, we get

$$\begin{aligned}
& y' = -\tfrac{2}{t}y + 4t \\
\implies \quad & y' + \tfrac{2}{t}y = 4t \\
\implies \quad & t^2\left(y' + \tfrac{2}{t}y\right) = t^2(4t) \\
\implies \quad & y't^2 + 2ty = 4t^3 \\
\implies \quad & (yt^2)' = 4t^3 \\
\implies \quad & yt^2 = t^4 + c, \qquad c \in \mathbb{R}. \\
\implies \quad & y = t + \frac{c}{t^2}, \qquad c \in \mathbb{R}.
\end{aligned}$$

Thus, the general solution to the differential equation is

$$y(t) = t + \frac{c}{t^2}, \qquad c \in \mathbb{R},\ t > 0.$$

If we now consider the initial condition $y(1) = -2$, then we see that $c = -3$ and hence the specific solution to the initial value problem is

$$y(t) = t - \frac{3}{t^2}, \qquad t > 0.$$

7.8 The Bernoulli Equation

In 1696, Jacob Bernoulli solved what is now known as the Bernoulli differential equation. This is a first-order *nonlinear* differential equation.

Definition 7.10. A *Bernoulli differential equation* is an equation of the form

$$u'(t) = p(t)u(t) + q(t)u^n(t), \tag{7.21}$$

where $p, q\colon I \to \mathbb{R}$ are two continuous functions defined on an open interval $I \subseteq \mathbb{R}$ and n is an integer strictly greater than one.

The Bernoulli equation is special in the following sense: it is a nonlinear differential equation that can be transformed into a linear differential equation. We now explain this method in more detail.

The Substitution Method: To transform the Bernoulli equation, which is nonlinear, into a linear equation, we use the substitution $v(t) = u(t)^{-(n-1)}$. Afterwards, we can solve the linear equation for v using the integrating factor method. The last step is to transform back to $u(t) = v(t)^{-\frac{1}{(n-1)}}$.

Divide the Bernoulli equation by $u(t)^n$ to obtain

$$\frac{u'(t)}{u(t)^n} = \frac{p(t)}{u(t)^{n-1}} + q(t).$$

Introduce the new unknown $v(t) = u(t)^{-(n-1)}$ and compute its derivative,

$$v'(t) = -(n-1)u(t)^{-n}u'(t) \qquad \Longrightarrow \qquad -\frac{v'(t)}{n-1} = \frac{u(t)'}{u(t)^n}.$$

If we substitute $v(t) = u(t)^{-(n-1)}$ and $-\frac{v'(t)}{n-1} = \frac{u(t)'}{u(t)^n}$ into the Bernoulli equation, we get

$$-\frac{v'(t)}{n-1} = p(t)v(t) + q(t)$$

which is a linear differential equation because it can be rearranged as

$$v'(t) = -(n-1)p(t)v(t) - (n-1)q(t).$$

Theorem 7.8. *Let $p, q\colon I \to \mathbb{R}$ be two continuous functions defined on an open interval $I \subseteq \mathbb{R}$ and let $n \geqslant 2$ be an integer. A function $u\colon J \to (0, +\infty)$ (respectively $u\colon J \to (-\infty, 0)$) is a solution to the Bernoulli differential equation*

$$u'(t) = p(t)u(t) + q(t)u^n(t)$$

if and only if the function $v(t) = u(t)^{-(n-1)}$ is a solution to the linear differential equation

$$v'(t) = -(n-1)p(t)v(t) - (n-1)q(t).$$

Remark 7.4. Note that Theorem 7.8 only allows us to find solutions to Bernoulli differential equations when the co-domain of the solution is either $(0, +\infty)$ or $(-\infty, 0)$. If there exist solutions whose range is not entirely contained in either $(0, +\infty)$ or $(-\infty, 0)$, then the method provided by Theorem 7.8 will not detect this solution. For instance, the zero function, i.e., the function $u\colon \mathbb{R} \to \mathbb{R}$ satisfying $u(t) = 0$ for all $t \in \mathbb{R}$, is a solution to every Bernoulli differential equation but completely overlooked by this method.

Corollary 7.5 (Maximal solutions to Bernoulli equations)**.** *Assume $p, q\colon I \to \mathbb{R}$ are continuous functions on an open interval $I \subseteq \mathbb{R}$ and consider the Bernoulli differential equation*

$$u'(t) = p(t)u(t) + q(t)u^n(t). \tag{7.22}$$

Let $v\colon I \to \mathbb{R}$ be a solution to the linear differential equation

$$v'(t) = -(n-1)p(t)v(t) - (n-1)q(t).$$

The set $\{t \in I : v(t) > 0\}$, if non-empty, is a disjoint union of open intervals $J^+_{max,1} \cup J^+_{max,2} \cup \dots$, and for each such interval $J^+_{max,i}$ we have:

(i) if n is even then the function $u\colon J^+_{max,i} \to (0,+\infty)$ defined as $u(t) = v(t)^{-\frac{1}{n-1}}$ is a maximal solution to (7.22) with maximal interval $J^+_{max,i}$;
(ii) if n is odd then the functions $u^+\colon J^+_{max,i} \to (0,+\infty)$ and $u^-\colon J^+_{max,i} \to (-\infty,0)$ defined as $u^\pm(t) = \pm v(t)^{-\frac{1}{n-1}}$ are maximal solutions to (7.22) with maximal interval $J^+_{max,i}$.

Similarly, the set $\{t \in \mathbb{R} : v(t) < 0\}$, if non-empty, is a disjoint union of open intervals $J^-_{max,1} \cup J^-_{max,2} \cup \dots$, and for each such interval $J^-_{max,i}$ we have:

(i) if n is even then the function $u\colon J^+_{max,i} \to (-\infty,0)$ defined as $u(t) = v(t)^{-\frac{1}{n-1}}$ is a maximal solution to the Bernoulli differential equation (7.22) with maximal interval $J^-_{max,i}$;
(ii) if n is odd then the interval $J^-_{max,i}$ supports no solutions to (7.22).

Aside from the trivial solution $u(t) = 0$ for all $t \in I$, the above describes all maximal solutions to he Bernoulli differential equation (7.22).

Example 7.18. Let us find every nonzero solution of the differential equation

$$y' = y + 2y^5.$$

This is a Bernoulli equation for $n = 5$. Divide the equation by y^5,

$$\frac{y'}{y^5} = \frac{1}{y^4} + 2.$$

Introduce the function $v = \frac{1}{y^4}$ and its derivative $v' = -4\left(\frac{y'}{y^5}\right)$, into the differential equation above,

$$-\frac{v'}{4} = v + 2 \quad \Longrightarrow \quad v' = -4v - 8 \quad \Longrightarrow \quad v' + 4v = -8.$$

The last equation is a linear differential equation for the function v. This equation can be solved using the integrating factor method. Multiply the equation by e^{4t}, then

$$e^{4t}v' + 4e^{4t}v = -8e^{4t} \quad \Longrightarrow \quad \left(e^{4t}v\right)' = -8e^{4t} \implies e^{4t}v = -2e^{4t} + c.$$

We obtain that $v = ce^{-4t} - 2$. Since $v = \frac{1}{y^4}$,

$$\frac{1}{y^4} = ce^{-4t} - 2 \quad \Longrightarrow \quad y(t) = \pm\frac{1}{(ce^{-4t} - 2)^{\frac{1}{4}}}.$$

Thus, for $c \leqslant 0$ we obtain no solution, but for each $c > 0$ we obtain two solutions,

$$y\colon \left(-\infty, \tfrac{1}{4}\ln\left(\tfrac{c}{2}\right)\right) \to (0, +\infty), \quad y(t) = \frac{1}{(ce^{-4t} - 2)^{\frac{1}{4}}},$$
$$y\colon \left(-\infty, \tfrac{1}{4}\ln\left(\tfrac{c}{2}\right)\right) \to (-\infty, 0), \quad y(t) = -\frac{1}{(ce^{-4t} - 2)^{\frac{1}{4}}}.$$

These are maximal solutions with maximal intervals $(-\infty, \frac{1}{4}\ln(\frac{c}{2}))$.

Example 7.19. Consider the Bernoulli equation

$$u'(t) + 2tu(t) = e^{t^2}u^2(t),$$

for the initial condition $u(0) = 2$. Divide both sides of the equation by $u^2(t)$ to obtain

$$\frac{u'(t)}{u^2(t)} + \frac{2t}{u(t)} = e^{t^2}.$$

By making the substitution $v(t) = \frac{1}{u(t)}$, we get the following linear differential equation for v:

$$v'(t) = 2tv(t) - e^{t^2},$$

for the initial condition $v(0) = \frac{1}{u(0)} = \frac{1}{2}$. In view of Corollary 7.4, the solution to this initial value problem is given by

$$v(t) = \underbrace{\tfrac{1}{2}e^{t^2} - \int_0^t e^{t^2 - s^2}e^{s^2}\,ds}_{v_0 e^{A(t)-A(t_0)} + \int_{t_0}^t e^{A(t)-A(s)}b(s)\,ds} = \tfrac{1}{2}e^{t^2} - te^{t^2}.$$

Reversing the substitution, using $u(t) = \frac{1}{v(t)}$, yields the expression for $u(t)$:

$$u(t) = \frac{1}{\frac{1}{2}e^{t^2} - te^{t^2}} = \frac{2e^{-t^2}}{1 - 2t}.$$

Thus, the solution to the initial value problem is

$$u\colon (-\infty, \tfrac{1}{2}) \to (0, +\infty), \quad u(t) = \frac{2e^{-t^2}}{1 - 2t}.$$

This is a maximal solution with maximal interval $(-\infty, \frac{1}{2})$.

Example 7.20. Consider the differential equation

$$\frac{dy}{dx} - \frac{3}{x}y = x^2y^4.$$

This is a Bernoulli differential equation, since in can be written in the standard Bernoulli form

$$\frac{dy}{dx} = p(x)y + q(x)y^n$$

with $p(x) = \frac{3}{x}$, $q(x) = x^2$, and $n = 4$. Make the substitution $v = y^{1-n} = y^{-3}$, so that

$$\frac{dv}{dx} = -3y^{-4}\frac{dy}{dx} \qquad \Longrightarrow \qquad y^{-4}\frac{dy}{dx} = -\frac{1}{3}\frac{dv}{dx}$$

Divide the original equation by y^4:

$$y^{-4}\frac{dy}{dx} - \frac{3}{x}y^{-3} = x^2$$

Substitute $v = y^{-3}$:

$$-\frac{1}{3}\frac{dv}{dx} - \frac{3}{x}v = x^2$$

Multiply through by -3:

$$\frac{dv}{dx} + \frac{9}{x}v = -3x^2$$

This is a linear ODE in v. The integrating factor is

$$e^{\int \frac{9}{x}dx} = x^9.$$

Multiply both sides by x^9, we get

$$x^9\frac{dv}{dx} + 9x^8v = -3x^{11} \qquad \Longrightarrow \qquad \frac{d}{dx}(x^9v) = -3x^{11}.$$

Integrate both sides,

$$x^9v = \int -3x^{11}dx = -\frac{3}{12}x^{12} + C = -\frac{1}{4}x^{12} + C,$$

and solve for v,

$$v = x^{-9}\left(-\frac{1}{4}x^{12} + C\right) = -\frac{1}{4}x^3 + Cx^{-9}.$$

Recall $v = y^{-3}$, so

$$y^{-3} = -\frac{1}{4}x^3 + Cx^{-9} \qquad \Longrightarrow \qquad y(x) = \left(-\frac{1}{4}x^3 + Cx^{-9}\right)^{-1/3}.$$

Final answer:

$$y(x) = \left(-\frac{1}{4}x^3 + Cx^{-9}\right)^{-1/3}$$

7.9 Second Order Linear Differential Equations

Definition 7.11. Let $I \subseteq \mathbb{R}$ be an open interval and let $a_0, a_1, b\colon I \to \mathbb{R}$ be continuous functions. A differential equation of the form

$$y''(t) \;=\; a_1(t)\, y'(t) + a_0(t)\, y(t) + b(t) \tag{7.23}$$

coefficients source function

is called a *second order linear differential equation*. The equation (7.23) above:

(i) is *homogeneous* if the source function $b(t) = 0$ for all $t \in I$;
(ii) is *inhomogeneous* if the source function $b(t) \neq 0$ for some $t \in I$;
(iii) has *constant coefficient* if $a_0(t) = a_0$ and $a_1(t) = a_1$ for some constants $a_0, a_1 \in \mathbb{R}$ and for all $t \in I$;
(iv) has *variable coefficient* if either $a_0(t)$ or $a_1(t)$ is a non-constant function on I;

Example 7.21. The famous *Schrödinger equation* in Quantum Mechanics, in one space dimension, stationary, is:

$$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi,$$

where ψ is the wave function of a particle (i.e., the probability density of finding the particle at the position x), m is the mass of the particle, $V(x)$ is the potential energy function, E is the total energy, and $\hbar$ is the reduced Planck constant (i.e., the Planck constant divided by 2π). It is an example of a second order homogeneous linear differential equation with variable coefficients.

Theorem 7.9 (Principal of Superposition)**.** *Let I be an open interval in $\mathbb{R}$, let $a_0, a_1, b_1, b_2\colon I \to \mathbb{R}$ be continuous functions and suppose the two functions $y_1, y_2\colon I \to \mathbb{R}$ are respective solutions to the two linear differential equations*

$$y''(t) = a_1(t)y'(t) + a_0(t)y(t) + b_1(t) \qquad \text{and} \qquad y''(t) = a_1(t)y'(t) + a_0(t)y(t) + b_2(t).$$

Then for any $c_1, c_2 \in \mathbb{R}$ the function $y\colon I \to \mathbb{R}$ defined by

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is a solution to the linear differential equation

$$y''(t) = a_1(t)y'(t) + a_0(t)y(t) + c_1 b_1(t) + c_2 b_2(t).$$

Theorem 7.10 (IVP Existence and Uniqueness Theorem)**.** *Let $I \subseteq \mathbb{R}$ be an open interval and let $a_0, a_1, b\colon I \to \mathbb{R}$ be continuous functions. For all $(t_0, y_0, y_1) \in I \times \mathbb{R} \times \mathbb{R}$ there is a unique solution $y\colon I \to \mathbb{R}$ of the initial value problem:*

$$y''(t) = a_1(t)y'(t) + a_0(t)y(t) + b(t) \qquad \text{and} \qquad y(t_0) = y_0, \quad y'(t_0) = y_1.$$

7.10 Second Order Homogeneous Linear Differential Equations

In this section we specifically discuss second order homogeneous linear differential equations, that is, equations of the form

$$y''(t) = a_1(t)y'(t) + a_0(t)y(t), \tag{7.24}$$

where $a_0, a_1 \colon I \to \mathbb{R}$ are continuous functions on an open interval $I \subseteq \mathbb{R}$.

Definition 7.12. Two solution $y_1, y_2 \colon I \to \mathbb{R}$ to the homogeneous equation (7.24) are:

(i) called *linearly dependent* if there exists a constant $c \in \mathbb{R}$ such that $y_1(t) = cy_2(t)$ holds for all $t \in I$.

(ii) called *linearly independent* if they are not linearly dependent.

A consequence of Theorem 7.10 is that the differential equation (7.24) has at least two linearly independent solutions. Indeed, let y_1 be a solution to the IVP with $y(t_0) = 1$ and $y'(t_0) = 1$, and y_2 a solution to the IVP with $y(t_0) = 1$ and $y'(t_0) = 2$; these two solutions cannot be linearly dependent, because if y_1 and y_2 were linearly dependent then

$$y_1(t_0) = y_2(t_0) = y_1'(t_0) = 1 \quad \Longrightarrow \quad y_2'(t_0) = 1.$$

Instead of 1 and 2, we could have chosen any other pair of distinct non-zero numbers. The next theorem tells us that it cannot be more than that and hence every second order homogeneous linear differential equation with variable coefficients has exactly two linearly independent solutions. Moreover, using the principle of superposition, all solutions can be produced via linear combinations of the two linearly independent ones.

Theorem 7.11. *Let I be an open interval of $\mathbb{R}$, $a_0, a_1 \colon I \to \mathbb{R}$ two continuous functions, and $y_1, y_2 \colon I \to \mathbb{R}$ two linearly independent solutions of the differential equation*

$$y''(t) = a_1(t)y'(t) + a_0(t)y(t)$$

Then the set of all solutions to the differential equation is given by the two parameter family of functions

$$y(t) = c_1y_1(t) + c_2y_2(t) \quad \textit{with} \quad c_1, c_2 \in \mathbb{R}. \tag{7.25}$$

By definition, the expression of (7.25) is called the general solution *of the differential equation.*

7.11 The Wronskian

We now introduce a function that provides important information about the linear dependency of two functions u_1, u_2 and is named after the polish scientist Josef Wronski.

Definition 7.13 (The Wronskian)**.** Let I be an open interval of $\mathbb{R}$. If $u_1, u_2 \colon I \to \mathbb{R}$

are two functions of class $C^1(I)$ then the function $W[u_1, u_2] : I \to \mathbb{R}$ defined by

$$W[u_1, u_2](t) = u_1(t)u_2'(t) - u_1'(t)u_2(t)$$

is called the *Wronskian* of u_1 and u_2.

Remark 7.5. The Wronskian is a determinant formed by arranging the functions u_1, u_2 and their derivatives into a matrix:

$$W[u_1, u_2](t) = \begin{vmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{vmatrix} = u_1(t)u_2'(t) - u_1'(t)u_2(t).$$

For this reason, the Wronskian is sometimes also called the *Wronskian determinant*.

Our next goal is to demonstrate Abel's Identity, which states that the Wronskian of two solutions of a second-order homogeneous linear differential equation satisfies its own first-order homogeneous linear differential equation. This leads to a useful description of all such Wronskians.

Suppose $y_1, y_2 \colon I \to \mathbb{R}$ are two solutions to the homogeneous linear differential equation (7.24) and consider their Wronskian $W[y_1, y_2] = y_1 y_2' - y_1' y_2$. If we take the derivative of the Wronskian then we see that

$$W[y_1, y_2]' = \frac{d}{dt} W[y_1, y_2] = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'} = y_1 y_2'' - y_1'' y_2.$$

Using that both y_1 and y_2 satisfy (7.24), we obtain

$$\begin{aligned} W[y_1, y_2]' = y_1 y_2'' - y_1'' y_2 &= y_1(a_1(t)y_2' + a_0(t)y_2) - (a_1(t)y_1' + a_0(t)y_1)y_2 \\ &= a_1(t)y_1y_2' + \cancel{a_0(t)y_1y_2} - a_1(t)y_1'y_2 + \cancel{a_0(t)y_1y_2} \\ &= a_1(t)y_1y_2' - a_1(t)y_1'y_2 \\ &= a_1(t)W[y_1, y_2]. \end{aligned}$$

In conclusion, we have found that the Wronskian satisfies the first order homogeneous linear differential equation

$$W[y_1, y_2]' = a_1(t)W[y_1, y_2]. \tag{7.26}$$

Since we already know how to solve such equations (cf. Theorem 7.7), we deduce that the Wronskian is equal to $W[y_1, y_2](t) = ce^{A_1(t)}$ where $A_1 \colon I \to \mathbb{R}$ is an antiderivative of $a_1(t)$ and c is a real number.

Theorem 7.12 (Abel's Identity)**.** *Let I be an open interval of $\mathbb{R}$ and $a_0, a_1 \colon I \to \mathbb{R}$ two continuous functions. If $y_1, y_2 \colon I \to \mathbb{R}$ are two solutions of the differential equation*

$$y''(t) = a_1(t)y'(t) + a_0(t)y(t)$$

then the Wronskian $W[y_1, y_2]$ satisfies the differential equation Eq. (7.26) and hence

$$W[y_1, y_2](t) = ce^{A_1(t)}$$

where $A_1 \colon I \to \mathbb{R}$ is an antiderivative of $a_1(t)$ and c is a real number.

If two functions u_1, u_2 are linearly dependent then their Wronskian $W[u_1, u_2]$ is constant equal to zero. While the reverse implication of this statement is not true in general, it follows from Abel's Identity that it is true when the two functions are solutions to a second order homogeneous linear differential equation.

7.12 Search for Linearly Independent Solutions to Homogeneous Equation

For first order homogeneous linear differential equations it is always possible to find solutions using the integrating factor method. When it comes to second order homogeneous linear differential equation, we know from Theorem 7.11 that it suffices to find two linearly independent solutions to obtain all solutions. Unfortunately, in general, there is no single method that allows us to find linearly independent solutions to (7.24). However, if we already know one non-zero solution then there is a method that allows us to turn this one solution into two linearly independent solutions. This method is based on Abel's identity and is explained next.

Method of Reduction of Order: If we already know one solution to a second order homogeneous linear differential equation then with the help of Abel's identity for the Wronskian, the task of finding a second solution reduces to solving a first order homogeneous linear differential equation. This method is called reduction of order, because we reduce a second-order equation to first-order equation when one solution is already known.

Suppose $y_1 \colon I \to \mathbb{R}$ is a non-zero solution to the homogeneous linear differential equation (7.24). We know from Theorem 7.12 that any other solution y_2 must satisfy

$$y_1 y_2' - y_1' y_2 = c e^{A_1(t)}$$

where $A_1 \colon I \to \mathbb{R}$ is an antiderivative of $a_1(t)$ and $c \in \mathbb{R}$. Note that for any $c \neq 0$, y_2 is a solution if and only if $c^{-1} y_2$ is a solution. Thus, replacing y_2 with $c^{-1} y_2$ if necessary, we can assume without loss of generality that $c = 1$ and hence

$$y_1 y_2' - y_1' y_2 = e^{A_1(t)}.$$

Since y_1 is known, the above is a first order inhomogenous linear differential equation in y_2, which we know how to solve due to Theorem 7.7. Indeed, if we invoke formula (7.18) with $c = 0$ then we get a second solution in the form

$$y_2(t) \;=\; y_1(t) \int_{t_0}^{t} \frac{e^{A_1(s)}}{y_1^2(s)}\, ds,$$

where $t_0 \in I$ is arbitrary. Since the function $\frac{e^{A_1(s)}}{y_1^2(s)}$ is non-zero, its antiderviatives are non-constant, which implies that y_2 and y_1 cannot be linearly dependent. The following theorem summarizes our findings.

Theorem 7.13. *Let I be an open interval of $\mathbb{R}$ and $a_0, a_1 \colon I \to \mathbb{R}$ two continuous functions. Suppose $y_1 \colon I \to \mathbb{R}$ is a non-zero solution to the homogeneous linear differential equation*

$$y''(t) = a_1(t)y'(t) + a_0(t)y(t)$$

then for any $t_0 \in I$ the function

$$y_2(t) \;=\; y_1(t) \int_{t_0}^{t} \frac{e^{A_1(s)}}{y_1^2(s)}\, ds,$$

is a second solution. Moreover, the two solutions y_1 and y_2 are linearly independent.

Example 7.22. Let us find all solutions to

$$t^2 y'' + 2ty' - 2y = 0, \quad t < 0,$$

given that $y_1(t) = t$ is a solution.

First, let us rearrange the differential equation to get

$$y'' = -\frac{2}{t}y' + \frac{2}{t^2}y.$$

When written in this form, we can see that this is a second order homogeneous linear differential equation with coefficients $a_1(t) = -\frac{2}{t}$ and $a_0(t) = \frac{2}{t^2}$. Thanks to Theorem 7.11, the set of all solutions to this differential equation is obtained by superpositions of two linearly independent solution. Since one solution is already given, it suffices to find a second solution that is linearly independent from the first. By Theorem 7.13, this second solution is given by

$$y_2(t) \;=\; y_1(t) \int_{t_0}^{t} \frac{e^{A_1(s)}}{y_1^2(s)}\, ds.$$

Using $t_0 = 1$, $y_1^2(t) = t^2$ and $A_1(t) = -2\ln(-t)$, where $A_1(t)$ denotes an antiderivative[2] of $a_1(t)$, we get

$$y_2(t) \;=\; t \int_{1}^{t} \frac{1}{s^4}\, ds \;=\; \frac{t}{3} - \frac{1}{3t^2}.$$

Thus, we have two linearly independent solutions $y_1(t) = t$ and $y_2(t) = \frac{t}{3} - \frac{1}{3t^2}$, and hence the general solution is given by linear combinations of these two solutions, which can be written as

$$y(t) = c_1 t + c_2 \frac{1}{t^2}, \qquad c_1, c_2 \in \mathbb{R}.$$

[2]Recall that the antiderivative of $\frac{1}{t}$ is given by

$$\int \frac{1}{t}\, dt = \begin{cases} \ln(t) + c & \text{if } t > 0, \\ \ln(-t) + c & \text{if } t < 0. \end{cases}$$