EXERCISE SHEET 9 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
April 14, 2025

1. Since

$$J_{\mathbf{u}}(x,y,z) = \begin{pmatrix} 2x & z^2\cos(yz^2) & 2yz\cos(yz^2) \\ -yx^{-2} & 1/x & 0 \end{pmatrix},$$

then

$$J_{\mathbf{u}}(1,0,1) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

2. We apply the chain rule to see that $J_{\mathbf{g}}(x,y,z) = J_{\mathbf{f}}(f(x,y,z)) \cdot J_{\mathbf{f}}(x,y,z)$ and

$$J_{\mathbf{f}}(x,y,z) = \begin{pmatrix} 0 & 2y & 0 \\ 0 & 0 & 2z \\ 2x & 0 & 0 \end{pmatrix}.$$

Thus,

$$J_{\mathbf{f}}(f(x,y,z)) = \begin{pmatrix} 0 & 2z^2 & 0 \\ 0 & 0 & 2x^2 \\ 2y^2 & 0 & 0 \end{pmatrix},$$

and

$$J_{\mathbf{g}}(x,y,z) = \begin{pmatrix} 0 & 2z^2 & 0 \\ 0 & 0 & 2x^2 \\ 2y^2 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2y & 0 \\ 0 & 0 & 2z \\ 2x & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4z^3 \\ 4x^3 & 0 & 0 \\ 0 & 4y^3 & 0 \end{pmatrix}.$$

3. We have $\{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, x^3 \le y \le x^{1/2}\} = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1, y^2 \le x \le y^{1/3}\}$.

4.

i) We have

$$\int_{-1}^{2} \left(\int_{0}^{1} \cos(x+y)\, dx \right) dy = \int_{-1}^{2} \Big[\sin(x+y) \Big]_{x=0}^{x=1} dy = \int_{-1}^{2} \big(\sin(1+y) - \sin(y) \big) dy$$
$$= \Big[-\cos(1+y) + \cos(y) \Big]_{-1}^{2} = 1 - \cos(1) + \cos(2) - \cos(3)\,.$$

ii) We have

$$\int_0^1 \left(\int_x^{2x} e^{x+y}\, dy \right) dx = \int_0^1 \left[e^{x+y} \right]_{y=x}^{y=2x} dx = \int_0^1 \left(e^{3x} - e^{2x} \right) dx = \left[\frac{1}{3}\, e^{3x} \right]_{x=0}^{x=1} - \left[\frac{1}{2}\, e^{2x} \right]_{x=0}^{x=1}$$
$$= \frac{1}{3}(e^3 - 1) - \frac{1}{2}(e^2 - 1) = \frac{1}{3}e^3 - \frac{1}{2}e^2 + \frac{1}{6}\,.$$

5.

i) Using Fubini's theorem, we have

$$\int_D \sqrt{x+y}\, dx\, dy = \int_0^1 \left(\int_0^2 \sqrt{x+y}\, dx \right) dy = \int_0^1 \left[\frac{2}{3}(x+y)^{3/2} \right]_{x=0}^{x=2} dy$$
$$= \int_0^1 \frac{2}{3} \left((2+y)^{3/2} - y^{3/2} \right) dy = \left[\frac{4}{15} \left((2+y)^{5/2} - y^{5/2} \right) \right]_0^1$$
$$= \frac{4}{15} \left(9\sqrt{3} - 4\sqrt{2} - 1 \right).$$

ii) Using Fubini's theorem, we have

$$\int_D x^2 y\, dx\, dy = \int_0^2 \left(\int_0^{x^2} x^2 y\, dy \right) dx = \int_0^2 \left[\frac{1}{2} x^2 y^2 \right]_{y=0}^{y=x^2} dx = \int_0^2 \frac{1}{2} x^6\, dx = \left[\frac{1}{14} x^7 \right]_0^2 = \frac{64}{7}\,.$$

iii) Using Fubini's theorem, we have

$$\int_D f(x,y)\, dx\, dy = -\int_0^1 \left(\int_0^x (x^2 - 2x - y^2 + 2y)\, dy \right) dx - \int_1^2 \left(\int_0^{2-x} (x^2 - 2x - y^2 + 2y)\, dy \right) dx$$
$$= -\int_0^1 \left[(x^2 - 2x)y - \frac{1}{3}y^3 + y^2 \right]_{y=0}^{y=x} dx - \int_1^2 \left[(x^2 - 2x)y - \frac{1}{3}y^3 + y^2 \right]_{y=0}^{y=2-x} dx$$
$$\overset{*}{=} -\int_0^1 \left(\frac{2}{3}x^3 - x^2 \right) dx - \int_1^2 \left(\frac{2}{3}(2-x)^3 - (2-x)^2 \right) dx$$
$$= -\left[\frac{1}{6}x^4 - \frac{1}{3}x^3 \right]_0^1 + \left[\frac{1}{6}(2-x)^4 - \frac{1}{3}(2-x)^3 \right]_1^2$$
$$= -\left(\frac{1}{6} - \frac{1}{3} \right) + \left(-\frac{1}{6} + \frac{1}{3} \right) = \frac{1}{3}\,.$$

For the step indicated by * we have rewritten the first term in the second integral as

$$(x^2 - 2x)(2-x) = -x(2-x)^2 = \big((2-x) - 2\big)(2-x)^2 = (2-x)^3 - 2(2-x)^2$$

to arrive at

$$(x^2 - 2x)(2-x) - \frac{1}{3}(2-x)^3 + (2-x)^2 = \frac{2}{3}(2-x)^3 - (2-x)^2$$

and thus avoiding the need to expand the polynomials.

6.

i) In the given order, we traverse D from bottom to top along horizontal lines. Reversing the order of integration is equivalent to traversing D from left to right along vertical lines. Thus x varies between 0 and 1 and y varies between 0 and x. We have

$$\int_0^1 \left(\int_y^1 e^{(x^2)}\, dx \right) dy = \int_0^1 \left(\int_0^x e^{(x^2)}\, dy \right) dx = \int_0^1 \left[y e^{(x^2)} \right]_{y=0}^{y=x} dx = \int_0^1 x e^{(x^2)}\, dx$$
$$= \left[\frac{1}{2} e^{(x^2)} \right]_0^1 = \frac{e-1}{2}.$$

ii) We must again reverse the order of integration to be able to calculate this integral, so we have

$$\int_0^1 \left(\int_{\sqrt[3]{y}}^1 \sqrt{1+x^4}\, dx \right) dy = \int_0^1 \left(\int_0^{x^3} \sqrt{1+x^4}\, dy \right) dx = \int_0^1 \left[y\sqrt{1+x^4} \right]_{y=0}^{y=x^3} dx$$
$$= \int_0^1 x^3 \sqrt{1+x^4}\, dx = \int_0^1 \frac{1}{4} 4x^3 (1+x^4)^{1/2}\, dx = \left[\frac{1}{6} \left(1+x^4\right)^{3/2} \right]_0^1 = \frac{1}{6}\left(2\sqrt{2}-1\right).$$

7. The points of the domain D satisfy

$$x \geq 0, \qquad -\sqrt{x} \leq y \leq \sqrt{x} \qquad \text{and} \qquad x-6 \leq y \leq x,$$

that is,

$$x \geq 0 \qquad \text{and} \qquad \max\{-\sqrt{x}, x-6\} \leq y \leq \min\{\sqrt{x}, x\}.$$

Let's determine the values of $x \geq 0$ such that $-\sqrt{x} = x-6$ or $\sqrt{x} = x$. For $x \geq 0$, we have

$$x-6 = -\sqrt{x} \iff (\sqrt{x})^2 + \sqrt{x} - 6 = 0 \iff (\sqrt{x}-2)(\sqrt{x}+3) = 0 \iff x = 4$$

and

$$\sqrt{x} = x \iff x \in \{0,1\}.$$

Let's consider the cases $x \in [0,1]$, $x \in [1,4]$ and $x \geq 4$.

For $x \in [0,1]$, we obtain $-\sqrt{x} \geq x-6$ and $\sqrt{x} \geq x$. Hence $-\sqrt{x} \leq y \leq x$.

For $x \in [1,4]$, we obtain $-\sqrt{x} \geq x-6$ and $\sqrt{x} \leq x$. Hence $-\sqrt{x} \leq y \leq \sqrt{x}$.

For $x \geq 4$, we obtain $-\sqrt{x} \leq x-6$ and $\sqrt{x} \leq x$. Where $x-6 \leq y \leq \sqrt{x}$. Such a y exists if and only if $x-6 \leq \sqrt{x}$. However, for $x \geq 4$,

$$x-6 = \sqrt{x} \iff (\sqrt{x})^2 - \sqrt{x} - 6 = 0 \iff (\sqrt{x}+2)(\sqrt{x}-3) = 0 \iff x = 9$$

and thus, we only need to consider $x \in [4,9]$ in the third case.

In summary,

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ -\sqrt{x} \leq y \leq x\} \cup \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 4,\ -\sqrt{x} \leq y \leq \sqrt{x}\}$$
$$\cup \{(x,y) \in \mathbb{R}^2 : 4 \leq x \leq 9,\ x-6 \leq y \leq \sqrt{x}\}.$$

The area of D is therefore

$$\iint\limits_D dx\,dy = \int_0^1 \left(\int_{-\sqrt{x}}^{x} dy \right) dx + \int_1^4 \left(\int_{-\sqrt{x}}^{\sqrt{x}} dy \right) dx + \int_4^9 \left(\int_{x-6}^{\sqrt{x}} dy \right) dx$$
$$= \int_0^1 \left(x + \sqrt{x}\right) dx + \int_1^4 2\sqrt{x}\, dx + \int_4^9 \left(\sqrt{x} - x + 6\right) dx$$
$$= \left[\frac{1}{2}x^2 + \frac{2}{3}x^{3/2} \right]_0^1 + \left[\frac{4}{3}x^{3/2} \right]_1^4 + \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 6x \right]_4^9$$
$$= \left(\frac{1}{2} + \frac{2}{3} \right) + \left(\frac{32}{3} - \frac{4}{3} \right) + \left(18 - \frac{81}{2} + 54 \right) - \left(\frac{16}{3} - 8 + 24 \right) = \frac{62}{3}.$$

8.

i) We have

$$\iint\limits_D y e^{y^2-4x} \underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}} = \int_0^2 \int_0^{\sqrt{8}} y e^{y^2-4x} \underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{x}} = \int_0^2 \int_0^{\sqrt{8}} \frac{\partial}{\partial y} \left(\frac{1}{2} e^{y^2-4x} \right) \underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{x}}$$
$$= \int_0^2 \left(\frac{1}{2} e^{y^2-4x} \right) \Big|_0^{\sqrt{8}} \underset{\cdot}{\mathrm{x}} = \frac{1}{2} \int_0^2 (e^{8-4x} - e^{-4x}) \underset{\cdot}{\mathrm{x}}$$
$$= \frac{1}{2} \left(-\frac{1}{4} e^{8-4x} + \frac{1}{4} e^{-4x} \right) \Big|_0^2 = \frac{1}{8}(e^8 + e^{-8} - 2).$$

ii) By solving the equation $1 - x^2 = x^2 - 3$, we see that the curves $y = 1 - x^2$, $y = x^2 - 3$ meet at the points $(\pm\sqrt{2}, -1)$, and for any $x \in [-\sqrt{2}, \sqrt{2}]$, we have $1 - x^2 \geq x^2 - 3$. Thus, $D = \{(x, y) \in \mathbb{R}^2\colon -\sqrt{2} \leq x \leq \sqrt{2},\ x^2 - 3 \leq y \leq 1 - x^2\}$. Therefore, we have

$$\iint\limits_D x(y-1) \underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}} = \int_{-\sqrt{2}}^{\sqrt{2}} x \int_{x^2-3}^{x^2-1} (y-1) \underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{x}} = \int_{-\sqrt{2}}^{\sqrt{2}} x \left(\frac{1}{2} y^2 - y \right) \Big|_{x^2-3}^{x^2-1} \underset{\cdot}{\mathrm{x}}$$
$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{1}{2} x(1-x^2)^2 - \frac{1}{2} x(x^2-3)^2 - x(1-x^2) + x(x^2-3) \right) \underset{\cdot}{\mathrm{x}}$$
$$= \int_{-\sqrt{2}}^{\sqrt{2}} (4x^3 - 8x) \underset{\cdot}{\mathrm{x}} = (x^4 - 4x^2) \Big|_{-\sqrt{2}}^{\sqrt{2}} = 0.$$

iii) As we did in ii), we find that $D = \{(x, y) \in \mathbb{R}^2\colon 0 \leq y \leq 2,\ 0 \leq x \leq 2\sqrt{y}\}$. Thus, we have

$$\iint\limits_D 5x^3 \cos(y^3) \underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}} = \int_0^2 \cos(y^3) \int_0^{2\sqrt{y}} 5x^3 \underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}} = \int_0^2 \cos(y^3) \left(\frac{5}{4} x^4 \right) \Big|_0^{2\sqrt{y}} \underset{\cdot}{\mathrm{y}}$$
$$= \int_0^2 20 y^2 \cos(y^3) \underset{\cdot}{\mathrm{y}} = \left(\frac{20}{3} \sin(y^3) \right) \Big|_0^2 = \frac{20}{8} \sin(8).$$

9. Solving $x^2 + z^2 = 4$, we get $z = \pm\sqrt{4 - x^2}$, thus the volume above the xy-plane is

$$\iint_D \sqrt{4 - x^2} \underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}}$$

and then the full volume V is

$$2\iint_D \sqrt{4-x^2}\,\mathrm{xy},$$

where D is the region determined by $x^2+y^2=4$. That is, $D=\{(x,y)\in\mathbb{R}^2\colon -2\le x\le 2,\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2}\}$. Therefore,

$$V=2\int_{-2}^{2}\sqrt{4-x^2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}1\mathrm{yx}=4\int_{-2}^{2}(4-x^2)\mathrm{x}=4\left(4x-\frac{x^3}{3}\right)\Big|_{-2}^{2}=\frac{128}{3}.$$

10. The equations of the straight lines delimiting the parallelogram D are given in Fig. 1.

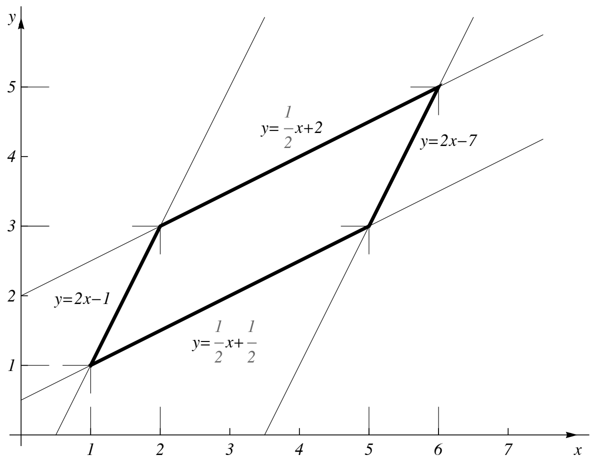

FIGURE 1

We divide the domain into three sub-domains by cutting along the vertical lines $x=2$ and $x=5$. So the area of the parallelogram is

$$\begin{aligned}\iint_D dx\,dy &= \int_1^2\left(\int_{\frac12x+\frac12}^{2x-1}dy\right)dx+\int_2^5\left(\int_{\frac12x+\frac12}^{\frac12x+2}dy\right)dx+\int_5^6\left(\int_{2x-7}^{\frac12x+2}dy\right)dx\\ &=\int_1^2\left(\frac32x-\frac32\right)dx+\int_2^5\frac32\,dx+\int_5^6\left(-\frac32x+9\right)dx\\ &=\frac32\left(\left(\frac12x^2-x\right)\Big|_1^2+x\Big|_2^5+\left(-\frac12x^2+6x\right)\Big|_5^6\right)=\frac32\cdot4=6\,.\end{aligned}$$

To calculate the area of D by a change of variables, we effectively try to transform D into a square—a shape of which we understand the area very well—and use the properties of this transformation to compute the area of D. As such, it is useful to re-express the equations of the lines as follows: $2x-y=1$, $2x-y=7$ and $x-2y=-1$, $x-2y=-4$. We then define a map H such that $(u,v)=H(x,y)$ with

$$\begin{cases}u=2x-y=H_1(x,y)\\ v=x-2y=H_2(x,y)\end{cases}$$

Now set $\widetilde{D} = [1,7] \times [-4,-1]$. We see that the image of $\mathring{D}$ under H is $\mathring{\widetilde{D}} =]1,7[\times]-4,-1[$ and $H \colon \mathring{D} \to \mathring{\widetilde{D}}$ is bijective. The Jacobian matrix of H and its determinant are

$$J_H(x,y) = \begin{pmatrix} \partial_x H_1(x,y) & \partial_y H_1(x,y) \\ \partial_x H_2(x,y) & \partial_y H_2(x,y)) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} \qquad \text{and} \qquad \det\big(J_H(x,y)\big) = -3\,.$$

Let $G = H^{-1} \colon \mathring{\widetilde{D}} \to \mathring{D}$ be the inverse transformation such that $(x,y) = G(u,v)$. The Jacobian of G is calculated from $J_H(x,y)$:

$$\det\big(J_G(u,v)\big) = \left[\frac{1}{\det\big(J_H(x,y)\big)}\right]_{(x,y)=G(u,v)} = -\frac{1}{3}\,.$$

The area of the parallelogram is then

$$\iint\limits_{D} dx\,dy = \iint\limits_{\mathring{D}} dx\,dy = \iint\limits_{\mathring{\widetilde{D}}} \big|\det\big(J_G(u,v)\big)\big|\,du\,dv = \int_1^7 \left(\int_{-4}^{-1} \frac{1}{3}\,du\right) dv = \frac{1}{3}\cdot 3 \cdot 6 = 6\,.$$

The result is obviously the same as before. But we have seen that it is faster to use an adequate change of variables.