Exercise Sheet 9

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
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Exercise 1. Given $D = \{(x, y, x) \in \mathbb{R}^3 : x \neq 0\}$. Let $\mathbf{u} : D \to \mathbb{R}^2$ be the function defined by

$$\mathbf{u}(x, y, z) = (x^2 + 1 + \sin(yz^2), y/x)^T.$$

The the Jocabian matrix $J_{\mathbf{u}}(1, 0, 1)$ is

A. $\begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$ B. $\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ D. $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Exercise 2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$\mathbf{f}(x, y, z) = (y^2, z^2, x^2)^T.$$

Then the Jocabian matrix of the composition $\mathbf{g} = \mathbf{f} \circ \mathbf{f}$ at (x, y, z) is

A. $J_{\mathbf{g}}(x, y, z) = \begin{pmatrix} 0 & 4y^2z & 0 \\ 0 & 0 & 4xz^2 \\ 4x^2y & 0 & 0 \end{pmatrix}$ B. $J_{\mathbf{g}}(x, y, z) = \begin{pmatrix} 0 & 4x^3 & 0 \\ 0 & 0 & 4y^3 \\ 4x^3 & 0 & 0 \end{pmatrix}$

C. $J_{\mathbf{g}}(x, y, z) = \begin{pmatrix} 0 & 0 & 4z^3 \\ 4x^3 & 0 & 0 \\ 0 & 4y^3 & 0 \end{pmatrix}$ D. $J_{\mathbf{g}}(x, y, z) = \begin{pmatrix} 0 & 0 & 4yz \\ 4zx & 0 & 0 \\ 0 & 4xy & 0 \end{pmatrix}$

Exercise 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then

A. $\int_0^1 \int_{x^3}^{\sqrt{x}} f(x, y) dy dx = \int_0^1 \int_{y^2}^{y^{1/3}} f(x, y) dx dy$ B. $\int_0^1 \int_{x^3}^{\sqrt{x}} f(x, y) dy dx = \int_0^1 \int_{y^{1/3}}^{y^2} f(x, y) dx dy$

C. $\int_0^1 \int_{x^3}^{\sqrt{x}} f(x, y) dy dx = \int_0^{\sqrt{x}} \int_{y^2}^{y^{1/3}} f(x, y) dx dy$ D. $\int_0^1 \int_{x^3}^{\sqrt{x}} f(x, y) dy dx = \int_0^1 \int_0^{y^{1/3}} f(x, y) dx dy$

Exercise 4. Calculate the following integrals:

i) $\displaystyle\int_{-1}^{2} \left(\int_0^1 \cos(x + y)\, dx \right) dy$ ii) $\displaystyle\int_0^1 \left(\int_x^{2x} e^{x+y}\, dy \right) dx.$

Exercise 5. Compute the double integral $\displaystyle\int_D f(x, y)\, dx\, dy$ in the following cases:

i) $f(x,y)=\sqrt{x+y}\,,\qquad D=\{(x,y):0\le x\le 2,\ 0\le y\le 1\}.$

ii) $f(x,y)=x^2y\,,\qquad D=\{(x,y):0\le y\le x^2,\ 0\le x\le 2\}.$

iii) $f(x,y)=|(x-y)(x+y-2)|\,,\qquad D=\{(x,y):0\le y\le x,\ x+y-2\le 0\}.$

Exercise 6. Calculate the following integrals:

i) $\displaystyle\int_0^1\left(\int_y^1 e^{x^2}\,dx\right)dy.$

ii) $\displaystyle\int_0^1\left(\int_{\sqrt[3]{y}}^1 \sqrt{1+x^4}\,dx\right)dy.$

Exercise 7. Calculate the area of the domain $D=\{(x,y)\colon y^2\le x,\ x-6\le y\le x\}$.

Exercise 8. Calculate the following double integrals:

i) $\displaystyle\iint_D ye^{y^2-4x}$xy, where $D=[0,2]\times[0,\sqrt{8}]$.

ii) $\displaystyle\iint_D x(y-1)$xy, where D is the region bounded by $y=1-x^2,\ y=x^2-3$.

iii) $\displaystyle\iint_D 5x^3\cos(y^3)$xy, where D is the region bounded by $y=2,\ y=\frac{x^2}{4}$ and the y-axis.

Exercise 9. Determine the volume of the region formed by the intersection of the two cylinders $x^2+y^2=4$ and $x^2+z^2=4$.

Exercise 10. Calculate the area of the parallelogram shown below first without and then with a change of variables. Does a change of variables seem useful to you in this case?